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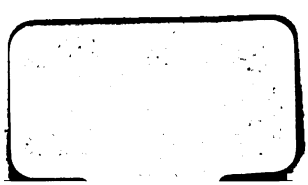

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ELEMENTS OF GEOMETRY:

CONSISTING OF

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OF

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CHIEFLY FROM

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WITH THE

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AND

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BY

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PROFESSOR OF MATHEMATICS, ETC., IN THE INSTITUTION.

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THE following Treatise on the Elements of Geometry is the second of a series which is to constitute a General Course of Mathematics for the use of the gentlemen cadets and the officers in the senior department of this Institution. The course, when completed, will comprehend the subjects whose titles are subjoined: I. Arithmetic and Algebra. II. Geometry. III. Plane and Spherical Trigonometry, with Mensuration. IV. Analytical Geometry and the Differential and Integral Calculus, with the Properties of the Conic Sections. V. Practical Astronomy and Geodesy. VI. The Principles of Mechanics; and VII. Physical Astronomy.

ROYAL MILITARY COLLEGE,
1842.

PREFACE.

GEOMETRY is the science in which are exhibited and proved various properties of finite magnitude under its three primary characteristics: — lines, planes, and solids, without regard to its physical nature.

It is well known from the testimony of ancient writers that many geometrical properties of magnitude had been discovered and demonstrated by learned men among the Greeks at a very early period; and it is scarcely probable that, almost immediately upon some important properties being discovered, there should not have been attempted an arrangement of the propositions according to their dependence upon one another and upon certain axioms, that is, upon such propositions as are considered self-evident.

But if any works of that nature existed previously to the age of Euclid, they must have been entirely superseded by the "Elements of Geometry" which bear the name of that mathematician, who is said to have been a disciple of the school of Plato, and to have cultivated the science at Alexandria between the years 323 and 284 B. C. This work is not only the most ancient treatise of geometry in existence, but the greater part of it has, down to the present moment, maintained the character of being the most proper introduction to the mathematical sciences. It is divided into thirteen books, of which the six first constitute what is called plane geometry: of these the four first exhibit the construction and principal properties of figures bounded by straight lines, together with some of the most elementary properties of circles: the fifth book treats of proportion among magnitudes, and in the sixth book the general theory of proportion is applied to geometrical plane figures. The seventh, eighth, and ninth books contain the properties of numbers; their greatest common measures and least common multiples; continued and mean proportionals. The tenth book relates to incommensurable quantities. The eleventh

and twelfth books treat of the intersections of planes, the properties of parallelepipeds, prisms, and pyramids, and of cylinders and cones: they contain also demonstrations of the ratios that circles and spheres, respectively, bear to one another. Lastly, the thirteenth book relates to equilateral and equiangular figures inscribed in circles, and to what are called the five regular bodies.

A fourteenth and a fifteenth book, both of which treat of the inscriptions of the five regular bodies in one another and in spheres, are frequently added to the Elements of Euclid; but they were written by Hypsicles of Alexandria. The same subjects have since been continued in a work which is sometimes called the sixteenth book of the Elements.

It will be for ever impossible now to decide whether the arrangements and demonstrations of the propositions in the thirteen books are entirely the work of Euclid; whether he merely arranged the propositions and rendered consistent the demonstrations which had been given before his time; or, finally, whether part of the work may not be ascribed to Theon of Alexandria, who lived in the fourth century of our era. All these opinions have been advanced; but Dr. Robert Simson, who was distinguished for his profound acquaintance with the works of the Greek geometers, after a comparison of the existing copies with one another and with the works of Archimedes and Apollonius, has arrived at the conclusion (Preface to his edition of the Elements) that "Theon, or whoever was the editor of the present Greek text, by adding some things, suppressing others, and mixing his own with Euclid's demonstrations, has changed many things to the worse." Dr. Simson's first edition was published in 1750; and in that, as well as in all that have subsequently appeared under his name, the inaccuracies which he supposes the ancient editors to have put in place of the genuine demonstrations of Euclid have been corrected: it is even probable that, at the same time, there were removed some errors which may have existed in the original text.

A work in which the purity of the ancient science has been so carefully maintained deserves to be, and in nearly all that regards plane geometry, excepting the fifth book it has been, ever since its publication, very generally used as a text-book for the study of this branch of science. In fact, the editions of Euclid's plane geometry which have since been published by other mathematicians differ from that of Simson almost wholly in the language being abbreviated.

But the extent to which mathematical science has been carried since the time of Euclid, and the numerous applica-

tions which have been made of its researches to subjects of a physical nature, have rendered it necessary, for the purpose of gaining time, that the generality of students should omit those portions of the ancient geometry which are not now immediately requisite for, or capable of such applications: it has also been found convenient, in the propositions, relating to rectilineal solids, which are still retained, to diminish the length of Euclid's demonstrations; and, in those which relate to figures of revolution, to adopt principles which, without being less founded in truth, have not the strictness of those which are employed by the Greek geometer. Whatever licence, however, may be allowed for the sake of procuring a facility of investigation in propositions of a complex nature, no doubt can exist that the first steps in geometry should consist of propositions in the demonstrations of which the inferences are strictly drawn from assumptions explicitly stated, and from self-evident truths; for only by researches so conducted can the mind acquire habits of accurately deducing conclusions from the premises of a subject. Such habits being formed, processes of investigation founded on less rigorous principles may afterwards without impropriety be employed.

A desire to retain, in the demonstrations of the propositions, the exact logic of Euclid, as far as it can be done without occupying to an inconvenient extent the time of the student, has led to the adoption, in the following work, of the four first books, the sixth and part of the geometry of planes from Simson's edition of the "Elements." The order of the propositions has been preserved in the four first and the sixth books, and in the demonstrations of the propositions very few changes have been made; but the fifth, and all the books which follow the sixth, have been omitted or only partially followed; and it will be proper here to state the reason for adopting a method different from that of Euclid, in treating the subject of proportion among geometrical magnitudes.

Magnitudes in the abstract, when compared with one another in respect of their extent, are divided into those which are commensurable and those which are not so: these two conditions are defined at the commencement of the tenth book of Euclid, and in that book, the properties of magnitudes under both conditions are proved in considerable detail. If geometrical propriety, therefore, were studied, it is evident that such propositions of the tenth book as do not depend upon those which come after the fifth, should have formed part of the latter book. But the investigations relating to

incommensurable magnitudes, when conducted in the manner pursued by Euclid, have a degree of intricacy which renders them unfit to form a part of elementary instruction; and, as it is generally supposed to be impossible, by reasonings purely geometrical, to treat the subject of proportion with the requisite simplicity and brevity, it has become a frequent practice to employ algebraic processes alone in the investigations relating to that subject.

The principle on which Euclid's theory of proportion is founded, is laid down in the fifth definition of the fifth book; and Mr. Ivory, the distinguished author of the "Tract on Proportion," which is used at this Institution, observes, in his preface to that tract, that the definition, "though perfectly general, comprehending quantities that are not, as well as those which are, commensurable, and easily applied in the course of investigation, at least to subjects in the elementary geometry, is, on the other hand, very remote from common apprehension: the conditions of it," he adds, "are complex, and not easily reconciled to our first and most natural ideas on the subject." On this account he substituted for that definition one which may easily be comprehended by the generality of students, and which is therefore advantageous in rendering the theory of proportion more satisfactory to the minds of those who are entering upon that part of the elements. As the adoption of this principle permits, at the same time, the elementary propositions relating to proportion to be demonstrated in a geometrical manner, that tract is, in the present edition, put in place of the fifth book of Euclid. The definition alluded to is the tenth of those which precede the propositions in the tract on proportion; and being, in strictness, applicable to the comparison of commensurable magnitudes only, it becomes necessary to show how, by an approximation which may be carried to any degree of accuracy, the ratio between magnitudes which are incommensurable can be obtained, and this is the subject of the propositions A, B, and D, and of the first and second general theorems in the tract: the propositions A, B, and C contain the substance of the four first propositions in Euclid's tenth book. For the sake, however, of learners, who, postponing the study of proportion among geometrical magnitudes, may require immediately the corresponding theorems relating to numbers, an algebraic investigation is given after the geometrical demonstration of each proposition: those propositions which relate to compound proportion are investigated by algebraic processes only.

The first proposition of the sixth book is that which enters

into the comparison of all plane rectilineal figures with respect to magnitude, and its proof depends upon the general definition of proportion. Therefore, for the sake of preserving the demonstration as it stands in the text of Simson, the fifth (or general) definition in Euclid's fifth book has been retained among the definitions in the tract on proportion: there is given, however, in addition, a demonstration of that proposition in conformity to the tenth definition above-mentioned. The thirty-third proposition of the sixth book is also demonstrated according to the manner of Euclid, and again in conformity to the same definition.

In order that the geometry of plane figures might not be interrupted, as it is in "Euclid," by the propositions concerning solids, there are given, in the next place, the necessary propositions relating to the magnitudes of circles, and to the ratios between their circumferences, and between their areas or surfaces. This part of the "Elements" is followed by a tract containing the principal problems which relate to the tracing of lines and geometrical figures on the ground, and to the determination of inaccessible distances without instruments: subjects of considerable utility in one branch of the military art.

The tract on the intersection of planes, and that on the properties of prisms and pyramids, remain nearly the same as in the editions which have hitherto been in use at this Institution: they consist of propositions taken from the eleventh and twelfth books of Euclid's "Elements," and of others which were added by Mr. Ivory, in order to render the geometry of solids bounded by plane surfaces more complete. But the investigations in the tract on circles, and in that on solid figures of revolution (cylinders, cones, and spheres), differ from those which, in the editions just mentioned, and in that of Dr. Simson, are employed in propositions relating to such figures: and, without departing from the spirit of the ancient geometry, an effort has been made to render this part of the "Elements" less tedious for the learner. It will therefore, in this place, be proper to show, in a general way, the manner in which the subject of figures of revolution was treated by the ancients.

The direct methods used in the elementary geometry to determine the magnitudes of plane figures bounded by straight lines, or of solids bounded by planes only, and to prove the relations between them, fail when it is required to investigate the magnitudes and relations of figures bounded by curve lines or surfaces; and the Greek geometers found it necessary to introduce into the "Elements" a principle

by which the investigations relating to polygons, pyramids and prisms might be rendered applicable to circles, and to solids of revolution. This principle consists in considering one magnitude to be equivalent to another, when it can be proved that they do not differ from each other, either in excess or defect, by any assignable quantity: also, that two ratios are equal to one another when it can be proved that, in one ratio, the antecedent is not to the consequent, as, in the other ratio, the antecedent is to any term which differs from the consequent, in excess or defect, by an assignable quantity. The processes used in consequence of the introduction of this principle constitute the "Method of Exhaustions;" it is supposed to have been invented by Euclid, but of this there is no proof; and, as it is not probable that the properties demonstrated by it were unknown before the time of Euclid and Archimedes, it may be inferred that the method is of higher antiquity than the age in which these matheticians lived.

The principle is applied by Euclid in the tenth proposition of the twelfth book, where it is demonstrated that a cylinder is equal in volume to three times the volume of a cone having an equal base and an equal altitude; and the process may be generalised in the following manner. Let the cylinder be represented by A, and the cone by B; and let there be inscribed in A a series of rectilineal figures (prisms), having altitudes equal to that of A, and continually increasing in volume till, at length, agreeably to the first proposition of the tenth book, there is obtained one (X) which differs, in defect, from A by a magnitude less than any magnitude by which A may be supposed to exceed the multiple of B: it will follow that X is greater than that multiple of B; but the altitudes of A and B being equal, and the base of X being less than that of A or B, X is less than that multiple of B (because it is equal to the same multiple of a pyramid inscribed in B); that is, X is both greater and less than the multiple of B, which is absurd. Therefore A is not greater than the multiple of B. Then, inverting the process, assuming that B exceeds the like submultiple of A by an assignable magnitude, and reasoning as before, it may be shown that A is not less than the multiple of B: hence the equivalence is inferred. It is evident, however, that this reasoning, instead of strictly proving the equivalence of the magnitudes, shows only that A does not differ from the assumed multiple of B, either in excess or defect, by a magnitude so great as the least that can be assigned. It does not follow that A may not differ from the multiple of B by a magnitude less than that by which it

differs from X: this difference is, indeed, supposed to be less than the least magnitude that can be assigned, but it is not nothing.

This is called the method of exhaustions, because there are supposed to be inscribed in a figure of revolution all the different rectilineal figures that can be so inscribed; or the number of such figures is exhausted.

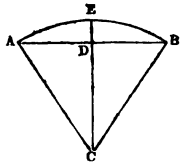
Again, Euclid's application of the method of exhaustions in finding the ratio between two figures of revolution (X and Y) of the same kind, consists in inscribing in Y a series of rectilineal figures continually increasing in magnitude, till, at length, there is found one (S) which shall differ from it by a magnitude less than that by which an assumed magnitude (M) differs, in defect, from Y; so that S is greater than M: there is then inscribed in X a rectilineal figure similar to S. Now, if the known ratio of the two inscribed rectilineal figures be represented by that of A to B, and it be required to prove that X and Y are to one another in the same ratio, let it be assumed first that A may have to B the same ratio as X has to M (which is less than Y); then it will follow, from equality of ratios, and because X is greater than the figure inscribed in it, that M is greater than S: but, as above, it is less, which is impossible; therefore A is not to B as X is to a magnitude less than Y. On inverting the process it may be proved in like manner that A is not to B, as X is to a magnitude greater than Y; and hence the equality of the ratios of A to B and of X to Y is inferred.

In strictness, however, it is only proved that A is to B as one of the figures (X) of revolution is to a magnitude which does not differ from the other figure (Y) either in excess or defect, by a quantity so great as the least that can be assigned. It does not follow that the magnitude may not differ from Y by a quantity less than that by which the figure inscribed in Y differs from Y: this difference is certainly less than that which is, by supposition, less than the least magnitude that can be assigned, but it is not nothing.

The method which, in this edition, is employed for determining the magnitudes and relations of figures formed by revolution, is founded upon the inscription, in a circle, of a regular polygon, each of whose sides is less than any line that can be assigned, and upon the description, about the circle, of a similar polygon, whose sides are also less than any assignable line. It is afterwards proved that the perimeters, and the areas of these polygons differ from the circumference, and from the area of the circle by lines and areas less than the least that can be assigned; also that the

volumes and surfaces of prisms or pyramids raised upon those polygons as bases, differ from the volume and surface of the cylinder or the cone raised upon the circle as a base, by volumes and surfaces less than the least that can be assigned: the required equivalence, or the required relation, is then at once inferred from a General Theorem in the tract on proportion. The reference is, however, made merely to avoid the necessity of stating the *reductio ad absurdum* in each proposition. The like process is employed for determining the volume and superficies of a sphere: the first is found from the cylinders formed by the revolution of rectangles inscribed within, and described about the circle by whose revolution the sphere is formed, and the other, from the surfaces produced by the revolution of a polygon inscribed within, and of one described about the circle.

Now Euclid, in proving (Prop. 2. B. 12.) by means of polygons inscribed in the circles, that circles are to one another as the squares of their diameters, does not state expressly that the sides of the inscribed polygons are less than the least assignable lines, but that the series of inscribed polygons is continued till there is obtained, in one of the circles, a polygon whose area shall differ from that of the circle by a magnitude less than that by which an assumed magnitude differs from the circle; this assumed magnitude may, however, differ from the circle by the least assignable magnitude; and therefore the last inscribed polygon may differ from the circle by a surface less than the least that can be assigned. But it is easy to show that a polygon inscribed in a circle cannot differ from the circle by a surface less than any that may be assigned, unless, at the same time, the sides of the polygon are, each, less than the least line that can be assigned. For, suppose AB to be a side of a polygon inscribed in the circle whose centre is C, and let CE be drawn perpendicularly to AB; consequently bisecting it in D. Then, if AB be an assignable line, its half, AD, will be assignable, and AD^2 will be an assignable surface: but $AD^2 = AC^2 - CD^2 = (AC+CD).(AC-CD) = (AC+CD).DE$;



therefore DE is an assignable line: but the segment AEB is greater than the triangle AEB, that is, greater than the rectangle AD.DE, which, being contained by lines not less than the least that can be assigned, is not less than the least surface that can be assigned; consequently, while AB is an assignable line, the polygon inscribed in the circle will differ from the circle by an area which is not less than the least

that can be assigned. Hence, in the demonstrations of Euclid, relating to the magnitudes of figures of revolution, it must be understood that the sides of the polygons inscribed in the circles are less than the least lines that can be assigned; and, if it can be shown that there may be inscribed in the circle a polygon of that kind, it will follow that the method employed in this work is in accordance with the spirit of the ancient geometry. Now it is a necessary corollary from the first proposition of Euclid's tenth book, that by continual bisections of the circumference of a circle, the whole circumference may, at length, be divided into parts, each of which has magnitude, and is less than the least line that can be assigned. Each part thus found is a curve line, for, otherwise, it must either be a point or a straight line: it cannot be a point, because a point has no magnitude, neither can it be a straight line, because, then, the circle would be a rectilineal polygon, which is absurd; therefore, granting that each part is a curve line, however small, it may be conceived that a straight line joining any two of the successive points of bisection, will cut off from the circle some segment, however small, and will be less than any assignable line: also, if a perpendicular be let fall from the centre of the circle upon such line, it will be less than a semidiameter of the circle. Thus, there may be conceived to be inscribed in a circle, a polygon whose sides are less than any lines that can be assigned, which, however, will possess the properties of any similar rectilineal figure whose sides are of assignable length.

The determination of the ratios which the surfaces, or the volumes, of solids of revolution bear to one another, might be investigated at length by processes corresponding to that by which the ratio of circles to one another is found; but this is scarcely necessary, after it has been shown that those surfaces, or volumes, are equivalent (the bases and altitudes of the figures being equal) to the areas of plane rectangles, or to the volumes of prisms. For it is easy then to infer that, whatever be the nature of the surfaces, or solids of revolution, they have the same ratios to one another as exist between the equivalent rectangles or prisms: in the present edition, these relations are, therefore, put in the form of corollaries to the propositions in which the surfaces, or volumes of the figures of revolution, are determined.

ELEMENTS

OF

GEOMETRY.

EUCLID. BOOK I.

DEFINITIONS.

I.

A POINT is that which has no parts, or which has no magnitude.

II.

A line is that which has length without breadth.

III.

The extremities of a line are points.

IV.

A straight line is that which lies evenly between its extreme points.

V.

A superficies is that which has only length and breadth.

VI.

The extremities of a superficies are lines.

VII.

A plane superficies is that in which any two points being taken, the straight line between them lies wholly in that superficies.

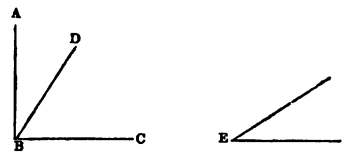
VIII.

" A plane angle is the inclination of two lines to one another " in a plane: the lines meeting, but not lying in the same " direction."

IX.

A plane rectilineal angle is the inclination of two straight lines to one another; the lines meeting, but not lying in the same direction.

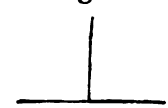
N. B. ' When several angles are at one point B, any one ' of them is expressed by three letters, of which the letter at ' the vertex of the angle, that ' is, at the point in which the ' straight lines containing the ' angle meet one another, is ' put between the two other ' letters; one of these two is



' somewhere upon one of those straight lines, and the other ' upon the other line: Thus the angle which is contained by ' the straight lines AB, CB, is named the angle ABC, or CBA; ' that which is contained by AB, DB is named the angle ABD, ' or DBA; and that which is contained by DB, CB is called ' the angle DBC, or CBD; but, if there be only one angle at ' a point, it may be expressed by a letter placed at that ' point; as the angle at E.'

X.

When a straight line standing on another straight line makes the adjacent angles equal to one another, each of the angles is called a right angle; and the straight line which stands on the other is called a perpendicular to it.



XI.

An obtuse angle is that which is greater than a right angle.

XII.

An acute angle is that which is less than a right angle.

XIII.

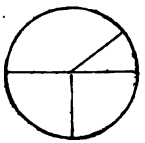
" A term or boundary is the extremity of any thing."

XIV.

A figure is that which is inclosed by one or more boundaries.

XV.

A circle is a plane figure contained by one line which is called the circumference, and is such that all straight lines drawn from a certain point within the figure to the circumference are equal to one another.



XVI.

This point is called the centre of the circle.

XVII.

A diameter of a circle is a straight line drawn through the centre, and terminated both ways by the circumference.

A semidiameter of a circle is now usually called the radius.

XVIII.

A semicircle is the figure contained by a diameter and the part of the circumference cut off by the diameter.

XIX.

"A segment of a circle is the figure contained by a straight "line, and the circumference it cuts off."

Any portion of the circumference of a circle is now usually designated an arc.

XX.

Rectilineal figures are those which are contained by straight lines.

XXI.

Trilateral figures, or triangles, by three straight lines.

XXII.

Quadrilateral, by four straight lines.

XXIII.

Multilateral figures, or polygons, by more than four straight lines.

XXIV.

Of three-sided figures, an equilateral triangle is that which has three equal sides.

XXV.

An isosceles triangle is that which has only two sides equal.

XXVI.

A scalene triangle is that which has three unequal sides.

XXVII.

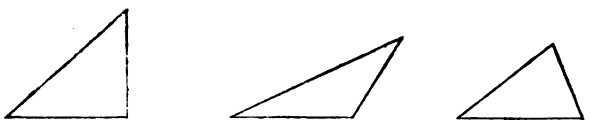
A right-angled triangle is that which has a right angle.

XXVIII.

An obtuse-angled triangle is that which has an obtuse angle.

XXIX.

An acute-angled triangle is that which has three acute angles.

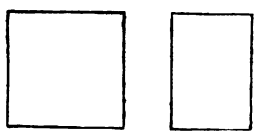


XXX.

Of four-sided figures, a square is that which has all its sides equal, and all its angles right angles.

XXXI.

An oblong is that which has all its angles right angles, but has not all its sides equal.

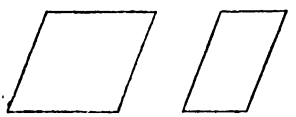


XXXII.

A rhombus is that which has all its sides equal, but its angles are not right angles.

XXXIII.

A rhomboid is that which has its opposite sides equal to one another, but all its sides are not equal, nor its angles right angles.



XXXIV.

All other four-sided figures besides these are called Trapeziums.

XXXV.

Parallel straight lines are such as are in the same plane, and which, being produced ever so far both ways, do not meet.

(A.)

A trapezium having two, only, of its sides parallel to one another is now called a trapezoid.

POSTULATES.

I.

LET it be granted that a straight line may be drawn from any one point to any other point.

II.

That a terminated straight line may be produced to any length in a straight line.

III.

That a circle may be described from any centre, at any distance from that centre.

IV. [Ax. XI.]

V.

And that if a straight line meets two straight lines, so as to make the two interior angles on the same side of it, taken together, less than two right angles, these straight lines, being continually produced, shall at length meet upon that side on which are the angles less than two right angles.

AXIOMS.

I.

THINGS which are equal to the same are equal to one another.

II.

If equals be added to equals, the wholes are equal.

III.

If equals be taken from equals, the remainders are equal.

IV.

If equals be added to unequals, the wholes are unequal.

V.

If equals be taken from unequals, the remainders are unequal.

VI.

Things which are doubles of the same are equal to one another.

VII.

Things which are halves of the same are equal to one another.

VIII.

Magnitudes which coincide with one another, that is, which exactly fill the same space, are equal to one another.

N.B. The term 'equivalent' is used in this work to denote equality of magnitude, in surface or volume, when figures (plane or solid) on being applied to one another will not coincide, or occupy the same space: the term 'equal' is used when, on being applied, they will coincide.

IX.

The whole is greater than its part.

X.

Two straight lines cannot inclose a space.

XI.

All right angles are equal to one another.

Remarks on the Definitions, &c. of the First Book.

The properties of geometrical magnitudes depend upon the definitions of the objects contemplated and upon certain abstractions of the mind, as position and straightness, which are among the qualities attributed to those objects: the investigations of the properties depend also on certain propositions which are assumed to be self-evident; and the impossibility of expressing abstract qualities otherwise than by negations is the cause that there is felt to be a want of precision in the language employed at the very commencement of the elements of geometry.

Euclid defines a point by saying that it has no parts or no magnitude (Def. I.): circumstances which may with equal correctness be predicated of non-entity; Professor Playfair designates it 'that which has position without magnitude,' which is nearly the definition said to have been given by Pythagoras (*a monad in position*), and which is still defective, since we have no idea of position, except as a part of space, which is, or may be, occupied by magnitude. Euclid's definition of a straight line, 'that which lies evenly, or equally, between its extreme points,' must be admitted to be very obscure. To lie evenly between the extreme points may signify that the line does not tend towards the parts of space on either of its sides; which is the sense of Plato's definition: and to lie equally between the extreme points may signify that the line divides the infinite space on the opposite sides of it into two equal parts. Archimedes defines a straight line to be the shortest distance between its extremities: and it is evident that none of these definitions has the precision which should characterise an elementary principle. The like observation may be made on Euclid's seventh definition, 'A plane superficies is that which lies evenly between its boundary lines.' That which Dr. Simson has substituted for it (in the text) is certainly more distinct; it has not, however, the propriety of corresponding to the previous definition of a straight line.

But the circumstance which has given most trouble to those geometers

who have attempted to improve the Elements of Euclid arises out of the properties of parallel lines. In the twenty-ninth proposition of the first book, the proof that 'if a straight line fall upon two parallel straight lines it will make the alternate angles equal to one another,' &c. is made to depend upon an assumption that, if a line falling upon two other lines makes with them the interior angles on the same side less than two right angles, those lines must meet if produced. Now this is evidently a theorem which ought to have been previously demonstrated, whereas Euclid has made it one of his postulates (Post. V.), and he thus leaves an elementary truth without direct support; for there exists no evidence of its certainty, except that which may be drawn from a converse of the twenty-eighth proposition of the first book, where it is shown that if a straight line falls upon two other straight lines, and makes the sum of the two interior angles upon the same side equal to two right angles, the two straight lines shall be parallel.

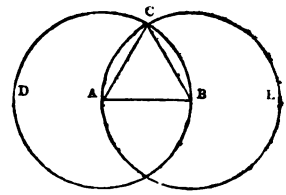
Numerous efforts have, at various times, been made to obtain a direct proof of this postulate, and also to change the assumption in the twenty-ninth proposition for one more simple; but it may be said that every such attempt which has been founded on the elementary geometry only contains, in some manner, the defect which it was intended to obviate. Professor Playfair and other mathematicians have recommended that, for the postulate, there should be substituted the axiom, 'Two lines which intersect one another cannot be parallel to a third;' which has the advantage over Euclid's postulate in simplicity only, while Dr. Simson has made the postulate itself an axiom, admitting at the same time that it is not a self-evident truth; and, in fact, in his note on the twenty-ninth proposition, he has given a demonstration of it in five theorems. This change of a postulate into an axiom, which cannot be said to carry in itself the evidence of its truth, and which therefore does not possess a character essential to all axioms, must be considered as unsatisfactory; and it will probably be admitted that, since every effort to remove the difficulty has hitherto been without success, the most legitimate course to be pursued is to leave the proposition in the form of a postulate. The place in which it ought properly to be introduced must, however, be understood to be immediately after the twenty-eighth proposition of the first book.

Euclid's fourth postulate has very properly been placed by Dr. Simson at the end of the axioms (Ax. XI.)

Dr. Simson considers that the eighth definition was intended to include an angle formed in a plane by the meeting of two curve lines, or of a straight line and a curve, as well as that formed by two straight lines, which is the subject of the ninth definition; and the two first kinds of plane angles being useless, Dr. Simson has distinguished the eighth by double inverted commas. For the reason last mentioned, the thirteenth definition is so marked: the nineteenth, properly belongs to the third book, where it is repeated.

PROPOSITION I. PROBLEM.

To describe an equilateral triangle upon a given finite straight line.



Let AB be the given straight line; it is required to describe an equilateral triangle upon it.

From the centre A, at the distance AB, describe (3. Postulate) the circle BCD, and from the centre B, at the distance BA, describe the circle ACE; and from the point C, in which the circles cut one another, draw the straight lines (1. Post.) CA, CB to the points A, B: ABC shall be an equilateral triangle.

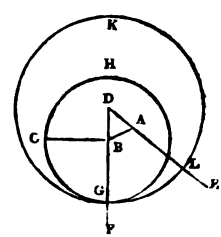
Because the point A is the centre of the circle BCD, AC is equal (15. Definition) to AB; and because the point B is the centre of the circle ACE, BC is equal to BA: but it has been proved that CA is equal to AB; therefore CA, CB are each of them equal to AB: but things which are equal to the same are equal to one another (1. Axiom); therefore CA is equal to CB; wherefore CA, AB, BC are equal to one another; and the triangle ABC is therefore equilateral, and it is described upon the given straight line AB. Which was required to be done.

In practice; instead of describing complete circles about A and B as centres, arcs or portions of the circumferences, of any convenient length, are described with a radius or semidiameter equal to AB, so as to intersect each other in C; and then the lines CA, CB are drawn, as in the text.

PROP. II. PROB.

From a given point to draw a straight line equal to a given straight line.

Let A be the given point, and BC the given straight line; it is required to draw from the point A a straight line equal to BC.



From the point A, to B draw (1. Post.) the straight line AB; and upon it describe (1. 1.) the equilateral triangle DAB, and produce (2. Post.) the straight lines DA, DB, to E and F: from the centre B, at the distance BC, describe (3. Post.) the circle CGH, and from the centre D, at the distance DG, describe the circle GKL: AL shall be equal to BC.

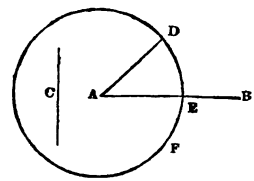
Because the point B is the centre of the circle CGH, BC is equal (15. Def.) to BG; and because D is the centre of the circle GKL, DL is equal to DG, and DA, DB, parts of them are equal (1. 1.): therefore the remainder AL is equal to the remainder (3 Ax.) BG: But it has been shown, that BC is equal to BG; wherefore AL and BC are each of them equal to BG; and things that are equal to the same are equal to one another; therefore the straight line AL is equal to BC. Wherefore from the given point A, a straight line AL has been drawn equal to the given straight line BC. Q. E. F.

In practice; from A as a centre, with a radius or semidiameter equal to the given line BC, describe an arc, or portion of the circumference of a circle in any convenient position; and draw a line from A to any point in the arc: the line so drawn will be equal to BC.

PROP. III. PROB.

FROM the greater of two given straight lines to cut off a part equal to the less.

Let AB and C be the two given straight lines, of which AB is the greater. It is required to cut off from AB, the greater, a part equal to C, the less.



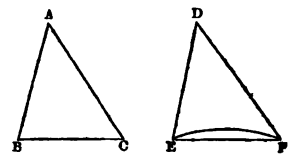
From the point A draw (2. 1.) the straight line AD equal to C; and from the centre A, and at the distance AD, describe (3. Post.) the circle DEF: then because A is the centre of the circle DEF, AE shall be equal to AD; but the straight line C is likewise equal to AD; whence AE and C are each of them equal to AD: wherefore the straight line AE is equal (1. Ax.) to C, and from AB, the greater of two straight lines, a part AE has been cut off equal to C, the less. Q. E. F.

PROP. IV. THEOREM.

IF two triangles have two sides of the one equal to two sides of the other, each to each; and have likewise the angles contained by those sides equal to one another; they shall likewise have their bases, or *third sides*, equal, and the two triangles shall be equal; and their other angles shall be equal, each to each, viz. those to which the equal sides are opposite.

Let ABC, DEF be two triangles, which have the two sides AB, AC equal to the two sides DE, DF, each to each, viz.

AB to DE, and AC to DF, and the angle BAC equal to the angle EDF; the base BC shall be equal to the base EF; and the triangle ABC to the triangle DEF; and the other angles to which the equal sides are opposite shall be equal, each to each, viz. the angle ABC to the angle DEF, and the angle ACB to DFE.



For if the triangle ABC be applied to DEF, so that the point A may be on D, and the straight line AB upon DE; the point B shall coincide with the point E, because AB is equal to DE; and AB coinciding with DE, AC shall coincide with DF, because the angle BAC is equal to the angle EDF; wherefore also the point C shall coincide with the point F, because the straight line AC is equal to DF: But the point B was proved to coincide with the point E; wherefore the base BC shall coincide with the base EF, because the point B coinciding with E, and C with F, if the base BC does not coincide with the base EF, two straight lines would inclose a space, which is impossible (10. Ax.) Therefore the base BC coincides with the base EF, and therefore is equal to it. Wherefore the whole triangle ABC coincides with the whole triangle DEF, and is equal to it; and the other angles of the one coincide with the remaining angles of the other, and are equal to them, viz. the angle ABC to the angle DEF, and the angle ACB to DFE. Therefore, if two triangles have two sides of the one equal to two sides of the other, each to each, and have likewise the angles contained by those sides equal to one another, their bases shall likewise be equal, and the triangles shall be equal, and the other angles to which the equal sides are opposite shall be equal, each to each. Which was to be demonstrated.

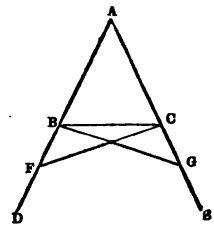
PROP. V. THEOR.

THE angles at the base of an isosceles triangle are equal to one another; and if the equal sides be produced, the angles upon the other side of the base shall be equal.

Let ABC be an isosceles triangle, of which the side AB is equal to AC, and let the straight lines AB, AC be produced to D and E; the angle ABC shall be equal to the angle ACB, and the angle CBD to the angle BCE.

In BD take any point F, and from AE, the greater, cut off AG equal (3. 1.) to AF, the less; also draw FC and GB.

Because AF is equal to AG, and AB to AC, the two sides FA, AC are equal to the two GA, AB, each to each; and they contain the angle FAG common to the two triangles AFC, AGB; therefore the base FC is equal (4. 1.) to the base GB, and the triangle AFC to the triangle AGB; and the remaining angles of the one are equal (4. 1.) to the remaining angles of the other, each to each, to which the equal sides are opposite, viz. the angle ACF to the angle ABG, and the angle AFC to the angle AGB: And because the whole AF is equal to the whole AG, of which the parts AB, AC are equal; the remainder BF is equal (3. Ax.) to the remainder CG; and FC was proved to be equal to GB; therefore the two sides BF, FC are equal to the two CG, GB, each to each; and the angle BFC was proved to be equal to the angle CGB; wherefore the triangles BFC and CGB are equal (4.1.), and also their remaining angles, each to each, to which the equal sides are opposite; therefore the angle FBC is equal to the angle GCB, and the angle BCF to the angle CBG: And since it has been demonstrated, that the whole angle ABG is equal to the whole ACF, the parts of which, the angles CBG, BCF, are also equal; therefore the remaining angle ABC is equal to the remaining angle ACB, which are the angles at the base of the triangle ABC: And it has also been proved that the angle FBC is equal to the angle GCB, which are the angles upon the other side of the base. Therefore the angles at the base, &c. Q. E. D.



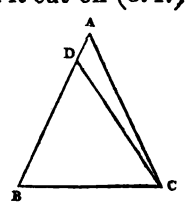
COROLLARY. Hence every equilateral triangle is also equiangular.

PROP. VI. THEOR.

IF two angles of a triangle be equal to one another the sides also which subtend, *or are opposite to*, the equal angles, shall be equal to one another.

Let ABC be a triangle having the angle ABC equal to the angle ACB; the side AB shall be equal to the side AC.

For, if AB be not equal to AC, one of them is greater than the other: Let AB be the greater; and from it cut off (3. 1.) DB equal to AC, the less, and join D, C: therefore, because in the triangles DBC, ACB, DB is equal to AC, and BC common to both, the two sides, DB, BC are equal to the two AC, CB, each to each; and the angle DBC is equal to the angle ACB; therefore the base DC is equal to the base AB, and the triangle



DBC is equal (4. 1.) to the triangle ACB, the less to the greater; which is absurd. Therefore AB is not unequal to AC, that is, it is equal to it. Wherefore, if two angles, &c. Q. E. D.

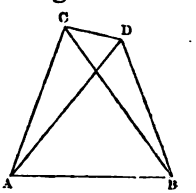
COR. Hence every equiangular triangle is also equilateral.

PROP. VII. THEOR.

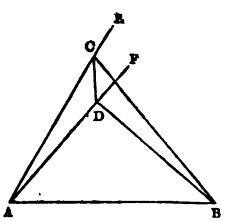
UPON the same base, and on the same side of it, there cannot be two triangles having their sides which are terminated in one extremity of the base equal to one another, and likewise those which are terminated in the other extremity.

If it be possible, let there be two triangles ACB, ADB, upon the same base AB, and upon the same side of it, which have their sides CA, DA, terminated in the extremity A of the base equal to one another, and likewise their sides CB, DB, that are terminated in B.

Join C, D; then, in the case in which the vertex of each of the triangles is on the exterior of the other triangle, because AC is equal to AD, the angle ACD is equal (5. 1.) to the angle ADC: But the angle ACD is greater than the angle BCD; therefore the angle ADC is greater also than BCD; much more then is the angle BDC greater than the angle BCD. Again, because CB is equal to DB, the angle BDC is equal (5. 1.) to the angle BCD; but it has been demonstrated to be greater than it; which is impossible.



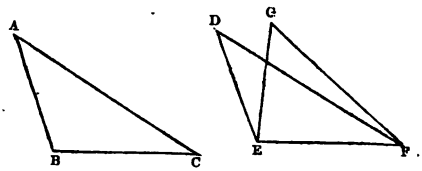
But if one of the vertices, as D, be within the other triangle, ACB; produce AC, AD to E, F; therefore, because AC is equal to AD in the triangle ACD, the angles ECD, FDC upon the other side of the base CD are equal (5. 1.) to one another, but the angle ECD is greater than the angle BCD; wherefore the angle FDC is likewise greater than BCD; much more then is the angle BDC greater than the angle BCD. Again, because CB is equal to BD, the angle BDC is equal (5. 1.) to the angle BCD; but BDC has been proved to be greaterthan the same BCD; which is impossible. The case in which the vertex of one triangle is upon a side of the other needs no demonstration.



Therefore, upon the same base, &c. Q. E. D.

PROP. VIII. THEOR.

If two triangles have two sides of the one equal to two sides of the other, each to each, and have likewise their bases equal; the angle which is contained by the two sides of the one shall be equal to the angle contained by the two sides equal to them of the other.



Let ABC, DEF be two triangles having the two sides AB, AC equal to the two sides DE, DF, each to each, viz. AB ot DE, and AC to DF; and also the base BC equal to the base EF. The angle BAC is equal to the angle EDF.

For, if the triangle ABC be applied to DEF, so that the point B be on E, and the straight line BC upon EF; the point C shall also coincide with the point F, because BC is equal to EF; therefore BC coinciding with EF; BA and AC shall coincide with ED and DF; for, if the base BC coincides with the base EF, but the sides BA, CA do not coincide with the sides ED, FD, but have a different situation as EG, FG, then, upon the same base EF, and upon the same side of it, there can be two triangles having their sides which are terminated in one extremity of the base equal to one another, and likewise their sides terminated in the other extremity: But this is impossible (7. 1.); therefore, if the base BC coincides with the base EF, the sides BA, AC cannot but coincide with the sides ED, DF; wherefore likewise the angle BAC coincides with the angle EDF, and is equal (8. Ax.) to it. Therefore if two triangles, &c. Q. E. D.

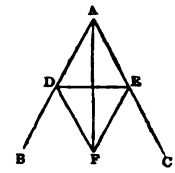
Cor. Hence, and from prop. IV., it follows that all the angles of the one triangle are equal to all the angles of the other, each to each, as in the conclusion of that proposition. Also that the two triangles are equal to one another in every respect.

PROP. IX. PROB.

To bisect a given rectilineal angle, that is, to divide it into two equal angles.

Let BAC be the given rectilineal angle, it is required to bisect it.

Take any point D in AB, and from AC cut off AE (3. 1.) equal to AD, join D, E, and upon it describe (1. 1.) an equilateral triangle DEF; then join A, F; the straight line AF shall bisect the angle BAC.



Because AD is equal to AE, and AF is common to the two triangles DAF, EAF; the two sides DA, AF, are equal to the two sides EA, AF, each to each: and the base DF is equal to the base EF; therefore the angle DAF is equal (8. 1.) to the angle EAF: wherefore the given rectilineal angle BAC is bisected by the straight line AF. Q. E. F.

In practice; from D and E as centres, with a radius equal to DE, or of any other convenient length, describe arcs intersecting one another in some point, as in F; the line AF will bisect the angle BAC; and the proof is the same as that which is given in the text.

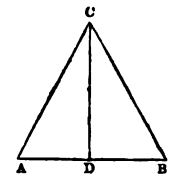
PROP. X. PROB.

To bisect a given finite straight line, that is, to divide it into two equal parts.

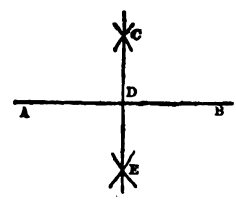
Let AB be the given straight line; it is required to divide it into two equal parts.

Describe (1. 1.) upon it an equilateral triangle ABC, and bisect (9. 1.) the angle ACB by the straight line CD. AB shall be cut into two equal parts in the point D.

Because AC is equal to CB, and CD common to the two triangles ACD, BCD; the two sides AC, CD are equal to BC, CD, each to each: and the angle ACD is equal to the angle BCD; therefore the base AD is equal (4. 1.) to the base BD, and the straight line AB is divided into two equal parts in the point D. Q. E. F.



In practice; from A and B as centres, with any convenient radius AC or BC greater than half the length of AB, describe arcs intersecting one another in C and E: draw CE intersecting AB in D; then AB will be bisected in D. For, suppose A and C, C and B, A and E, B and E, to be joined; then in the triangles ACE, BCE, AC is equal to CB, CE is common, and AE is equal to BE; therefore the angle ACE is equal to BCE (8. 1). Again, in the triangles ACD, BCD, AC is equal to BC, CD is common, and the angle ACD is equal to BCD; therefore, AD is equal to DB (4. 1.). Thus AB is bisected in D.

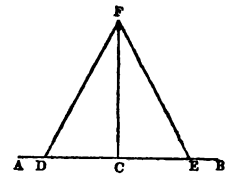


PROP. XI. PROB.

To draw a straight line at right angles to a given straight line, from a given point in the same.

Let AB be a given straight line, and C a point given in it; it is required to draw a straight line from the point C at right angles to AB.

Take any point D in AC, and (3. 1.) make CE equal to CD, and upon DE describe (1. 1.) the equilateral triangle DFE, and join F, C, the straight line FC drawn from the given point C shall be at right angles to the given straight line AB.

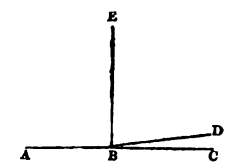


Because DC is equal to CE, and FC common to the two triangles DCF, ECF; the two sides DC, CF are equal to the two EC, CF, each to each; and the base DF is equal to the base EF; therefore the angle DCF is equal (8. 1.) to the angle ECF; and they are adjacent angles. But when the adjacent angles which one straight line makes with another straight line are equal to one another, each of them is called (10. Def.) a right angle; therefore each of the angles DCF, ECF is a right angle. Wherefore, from the given point C, in the given straight line AB, CF has been drawn at right angles to AB. Q. E. F.

In practice; having made CE equal to CD, from D and E as centres, with a radius equal to DE, or of any other convenient length greater than DC, describe arcs intersecting one another in some point, as F: the line CF is the perpendicular required; and the demonstration is the same as that which is given in the text.

Cor. By the help of this problem, it may be demonstrated, that two straight lines cannot have a common segment.

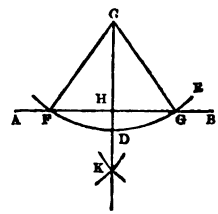
If it be possible, let the two straight lines ABC, ABD have the segment AB common to both of them. From the point B draw BE at right angles to AB; and because ABC is a straight line, the angle CBE is equal (10. Def.) to the angle EBA; in the same manner, because ABD is a straight line, the angle DBE is equal to the angle EBA; wherefore the angle DBE is equal to the angle CBE, the less to the greater; which is impossible; therefore two straight lines cannot have a common segment.



PROP. XII. PROB.

To draw a straight line perpendicular to a given straight line of an unlimited length, from a given point on its exterior.

Let AB be the given straight line, which may be produced to any length both ways, and let C be a point on its exterior. It is required to draw a straight line perpendicular to AB from the point C.

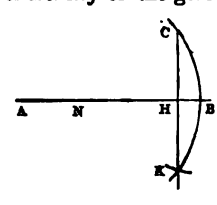


Take any point D upon the other side of AB, and from the centre C, at the distance CD, describe (3. Post.) the circle EGF meeting AB in F and G; bisect (10. 1.) FG in H, and draw CF, CH, CG; the straight line CH, drawn from the given point C, shall be perpendicular to the given straight line AB.

Because FH is equal to HG, and HC common to the two triangles FHC, GHC, the two sides FH, HC are equal to the two GH, HC, each to each; and the base CF is equal (15. Def.) to the base CG; therefore the angle CHF is equal (8. 1.) to the angle CHG; and they are adjacent angles: but when a straight line standing on another straight line makes the adjacent angles equal to one another, each of them is a right angle; and the straight line which stands upon the other is called a perpendicular to it; therefore from the given point C a perpendicular CH has been drawn to the given straight line AB. Q. E. F.

In practice; having described the arc FG, as in the text, from F and G as centres, with the same, or any convenient radius greater than half of FG, describe arcs intersecting each other in K: a line drawn from C to K will be perpendicular to AB. For imagine F, K and G, K to be joined; then, in the triangles FCK and GCK, CF is equal to CG, CK is common, and FK is equal to GK; therefore the angle FCK is equal to GCK (8. 1.). Again, in the triangles FCH and GCH, CF is equal to CG, CH is common, and the angle FCH is equal to GCH; consequently (4. 1.) the angle CHF is equal to CHG. Therefore these are right angles, and CH is perpendicular to AB.

When the given point C is nearly opposite one extremity of the given line. From any convenient point N in the line, and with a radius equal to NC, describe an arc, as CK, intersecting AB in any point B: from B as a centre, with a radius equal to a straight line imagined to be drawn from B to C, describe an arc intersecting the former arc in K, and join C, K. The line CK will be perpendicular to AB. For in the triangles NCB and NKB, the straight line BC is equal to the straight line BK, NC is equal to NK, being radii of the same circle, and NB is common; therefore the angle CNB is equal to KNB (8. 1.) Again, in the triangles NCH, NKH, NC



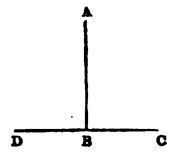
is equal to NK, NH is common, and the angle CNH is equal to KNH; therefore (4. 1.) the angle NHC is equal to NHK, and each of these is a right angle.

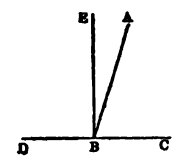
PROP. XIII. THEOR.

THE angles which one straight line makes with another upon one side of it are either two right angles, or are together equal to two right angles.

Let the straight line AB make with CD, upon one side of it, the angles CBA, ABD: these shall either be two right angles, or shall together be equal to two right angles.

For if the angle CBA be equal to ABD, each of them is a right (10. Def.) angle; but if not, from the point B draw BE at right angles (11. 1.) to CD; therefore the angles CBE, EBD are two right angles (10. Def.); and because the angle CBE is equal to the two angles CBA, ABE together; add the angle EBD to each of these equals; therefore the angles CBE, EBD are equal (2. Ax.) to the three angles CBA, ABE, EBD. Again, because the angle DBA is equal to the two angles DBE, EBA, add to each of these equals the angle ABC, therefore the angles DBA, ABC are equal to the three angles DBE, EBA, ABC: but the angles CBE, EBD have been demonstrated to be equal to the same three angles; and things that are equal to the same thing are equal (1. Ax.) to one another; therefore the angles CBE, EBD are equal to the angles DBA, ABC: but CBE, EBD are two right angles; therefore DBA, ABC are together equal to two right angles. Wherefore, when a straight line, &c. Q. E. D.





COR. 1. From this it is manifest, that if two straight lines cut one another, the angles which they make at the point where they cut, are together equal to four right angles.

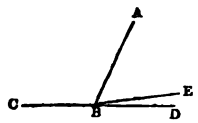
COR. 2. And, consequently, that all the angles made by any number of lines meeting in one point, are together equal to four right angles.

PROP. XIV. THEOR.

IF, at a point in a straight line, two other straight lines, upon the opposite sides of it, make the adjacent angles together equal to two right angles, these two

straight lines shall be in one and the same straight line.

At the point B in the straight line AB, let the two straight lines BC, BD upon the opposite sides of AB make the adjacent angles ABC, ABD equal together to two right angles, BD shall be in the same straight line with CB.



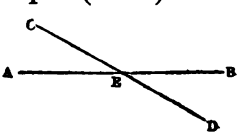
For, if BD be not in the same straight line with CB, let BE be in the same straight line with it; therefore, because the straight line AB makes angles with the straight line CBE, upon one side of it, the angles ABC, ABE are together equal (13. 1.) to two right angles; but the angles ABC, ABD are likewise together equal to two right angles; therefore the angles CBA, ABE are equal to the angles CBA, ABD: Take away the common angle ABC, and the remaining angle ABE is equal (3. Ax.) to the remaining angle ABD, the less to the greater, which is impossible; therefore BE is not in the same straight line with BC. And, in like manner, it may be demonstrated, that no other can be in the same straight line with it but BD, which therefore is in the same straight line with CB. Wherefore, if at a point, &c. Q. E. D.

PROP. XV. THEOR.

If two straight lines cut one another, the vertical, *or opposite*, angles shall be equal.

Let the two straight lines AB, CD cut one another in the point E; the angle AEC shall be equal to the angle DEB, and CEB to AED.

Because the straight line AE makes with CD the angles CEA, AED, these angles are together equal (13. 1.) to two right angles. Again, because the straight line DE makes with AB the angles AED, DEB, these also are together equal (13.1.) to two right angles; and CEA, AED have been demonstrated to be equal to two right angles; wherefore, the angles CEA, AED are equal to the angles AED, DEB. Take away the common angle AED, and the remaining angle CEA is equal (3. Ax.) to the remaining angle DEB. In the same manner it can be demonstrated that the angles CEB, AED are equal. Therefore, if two straight lines, &c. Q. E. D.

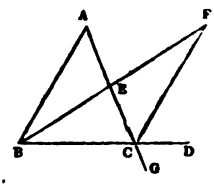


PROP. XVI. THEOR.

If one side of a triangle be produced, the exterior angle is greater than either of the interior opposite angles.

Let ABC be a triangle, and let its side BC be produced to D, the exterior angle ACD is greater than either of the interior opposite angles CBA, BAC.

Bisect (10. 1.) AC in E, join B, E, and produce BE to F, and make EF equal to BE; join also F, C, and produce AC to G.



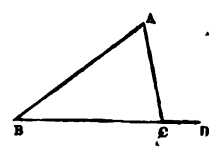
Because AE is equal to EC, and BE to EF; AE, EB are equal to CE, EF, each to each; and the angle AEB is equal (15. 1.) to the angle CEF, because they are opposite vertical angles; therefore the base AB is equal (4. 1.) to the base CF, and the triangle AEB to the triangle CEF, and the remaining angles to the remaining angles, each to each, to which the equal sides are opposite; wherefore the angle BAE is equal to the angle ECF: but the angle ECD is greater than the angle ECF; therefore the angle ACD is greater than BAE: In the same manner, if the side BC be bisected, it may be demonstrated that the angle BCG, that is (15. 1.), the angle ACD, is greater than the angle ABC. Therefore, if one side, &c. Q. E. D.

PROP. XVII. THEOR.

Any two angles of a triangle are together less than two right angles.

Let ABC be any triangle; any two of its angles together are less than two right angles.

Produce BC to D; and because ACD is the exterior angle of the triangle ABC, ACD is greater (16. 1.) than the interior and opposite angle ABC: to each of these add the angle ACB; therefore the angles ACD, ACB are greater than the angles ABC, ACB; but ACD, ACB are together equal (13. 1.) to two right angles; therefore the angles ABC, BCA are less than two right angles. In like manner, it may be de-



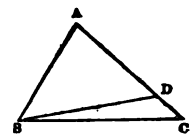
monstrated that BAC, ACB, as also CAB, ABC, are less than two right angles. Therefore any two angles, &c. Q. E. D.

PROP. XVIII. THEOR.

THE greater side of every triangle is opposite to the greater angle.

Let ABC be a triangle, of which the side AC is greater than the side AB; the angle ABC is also greater than the angle BCA.

Because AC is greater than AB, make (3. 1.) AD equal to AB, and join B, D: and because ADB is the exterior angle of the triangle BDC, it is greater (16. 1.) than the interior and opposite angle DCB; but ADB is equal (5. 1.) to ABD, because the side AB is equal to the side AD; therefore the angle ABD is likewise greater than the angle ACB. Wherefore much more is the angle ABC greater than ACB. Therefore the greater side, &c. Q. E. D.



PROP. XIX. THEOR.

THE greater angle of every triangle is subtended by the greater side, or *has the greater side opposite to it.*

Let ABC be a triangle, of which the angle ABC is greater than the angle BCA: the side AC is likewise greater than the side AB.

For, if it be not greater, AC must either be equal to AB, or less than it; it is not equal, because then the angle ABC would be equal (5. 1.) to the angle ACB: but it is not; therefore AC is not equal to AB: neither is it less; because then the angle ABC would be less (18. 1.) than the angle ACB; but it is not; therefore the side AC is not less than AB; and it has been shown that it is not equal to AB; therefore AC is greater than AB. Wherefore the greater angle, &c. Q. E. D.

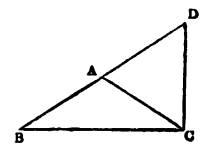
PROP. XX. THEOR.

ANY two sides of a triangle are together greater than the third side.

Let ABC be a triangle; any two sides of it together, are greater than the third side, viz. the sides BA, AC greater than the side BC; and AB, BC greater than AC; and BC, CA greater than AB.

Produce BA indefinitely, and make (3. 1.) AD equal to AC; and join D, C.

Because DA is equal to AC, the angle ADC is likewise equal (5. 1.) to ACD; but the angle BCD is greater than the angle ACD; therefore the angle BCD is greater than the angle ADC: and because the angle BCD of the triangle DCB is greater than its angle BDC, and that the greater angle (19. 1.) is opposite to the greater side, therefore the side DB is greater than the side BC: but DB is equal to BA and AC; therefore the sides BA, AC are greater than BC. In the same manner it may be demonstrated that the sides AB, BC are greater than CA, and BC, CA greater than AB. Therefore any two sides, &c. Q. E. D.

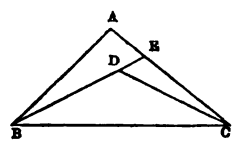


PROP. XXI. THEOR.

If from the ends of the side of a triangle there be drawn two straight lines to a point within the triangle, these shall be less than the two other sides of the triangle, but shall contain a greater angle.

Let the two straight lines BD, CD be drawn from B, C, the ends of the side BC of the triangle ABC, to the point D within it; BD and DC are less than the two other sides BA, AC of the triangle, but contain an angle BDC greater than the angle BAC.

Produce BD to E; and because two sides of a triangle are greater (20. 1.) than the third side, the two sides BA, AE of the triangle ABE are greater than BE. To each of these add EC; therefore the sides BA, AC are greater than BE, EC: Again, because the two sides CE, ED of the triangle CED are greater than CD, add DB to each of these; therefore the sides CE, EB are greater than CD, DB; but it has been shown that BA, AC are greater than BE, EC, much more then are BA, AC greater than BD, DC.



Again, because the exterior angle of a triangle is greater (16. 1.) than the interior and opposite angle, the exterior angle BDC of the triangle CDE is greater than CED; for the same

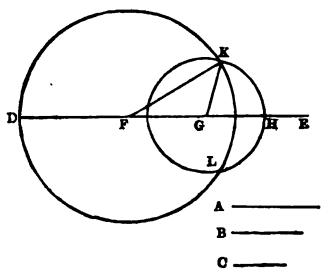
reason, the exterior angle CEB of the triangle ABE is greater than BAC; and it has been demonstrated that the angle BDC is greater than the angle CEB; much more then is the angle BDC greater than the angle BAC. Therefore, if from the ends of, &c. Q. E. D.

PROP. XXII. PROB.

To make a triangle of which the sides shall be equal to three given straight lines, but any two whatever of these must be greater than the third (20. 1.)

Let A, B, C be the three given straight lines of which any two whatever are greater than the third, viz. A and B greater than C; A and C greater than B; and B and C than A. It is required to make a triangle of which the *sides* shall be equal to A, B, C, each to each.

Take a straight line DE terminated at the point D, but unlimited towards E, and make (3. 1.) DF equal to A, FG to B, and GH equal to C; and from the centre F, at the distance FD, describe (3. Post.) the circle DKL; and from the centre G, at the distance GH, describe (3. Post.) another circle, HLK; and draw KF, KG; the triangle KFG has its sides equal to the three straight lines A, B, C.



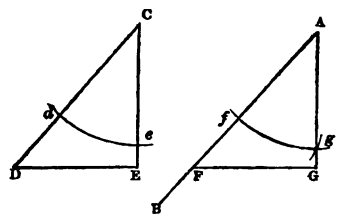
Because the point F is the centre of the circle DKL, FD is equal (15. Def.) to FK; but FD is equal to the straight line A; therefore FK is equal to A: Again, because G is the centre of the circle LKH, GH is equal to GK; but GH is equal to C; therefore also GK is equal to C; and FG is equal to B; therefore the three straight lines KF, FG, GK, are equal to the three A, B, C: And therefore the triangle KFG has its three sides KF, FG, GK equal to the three given straight lines, A, B, C. Q. E. F.

In practice; instead of describing complete circles about F and G as centres, arcs of circles, of any convenient length, are described with radii equal to the lines A and C respectively, about the same points as centres, so as to intersect each other in K.

PROP. XXIII. PROB.

At a given point in a given straight line, to make a rectilineal angle equal to a given rectilineal angle.

Let AB be the given straight line, and A the given point in it, and DCE the given rectilineal angle; it is required to make an angle at the given point A in the given straight line AB, that shall be equal to the given rectilineal angle DCE.



Take in CD, CE any points D, E, and join D, E; and make (22. 1.) the triangle AFG, the sides of which shall be equal to the three straight lines CD, DE, EC, so that CD be equal to AF, CE to AG, and DE to FG; and because DC, CE are equal to FA, AG, each to each, and the base DE to the base FG; the angle DCE is equal (8. 1.) to the angle FAG. Therefore at the given point A in the given straight line AB, the angle FAG is made equal to the given rectilineal angle DCE. Q. E. F.

In practice; from C as a centre, with any convenient radius as cd, describe the arc de; and from A as a centre with an equal radius describe the arc fg. Again, from f as a centre, with a radius equal to a straight line supposed to be drawn from d to e, describe an arc intersecting the arc fg in g, and draw a line from A through the point g: the angle BAg is equal to DCE. For cd and ce are equal to Af and Ag, and the straight line de to the straight line fg; therefore (8. 1.) the angle dce is equal to fAg.

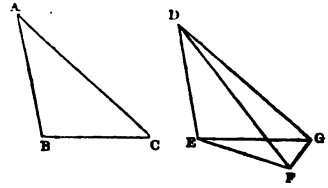
PROP. XXIV. THEOR.

If two triangles have two sides of the one equal to two sides of the other, each to each, but the angle contained by the two sides of one of them greater than the angle contained by the two sides equal to them of the other; the base of that which has the greater angle shall be greater than the base of the other.

Let ABC, DEF be two triangles which have the two sides AB, AC equal to the two DE, DF, each to each, viz. AB equal to DE, and AC to DF; but the angle BAC greater than the angle EDF; the base BC is also greater than the base EF.

Of the two sides DE, DF, let DE be the side which is not greater than the other, and at the point D, in the straight line DE, make (23. 1.) the angle EDG equal to the angle BAC; and made DG equal (3. 1.) to AC or DF, and draw EG, GF.

Because AB is equal to DE, and AC to DG, the two sides BA, AC are equal to the two ED, DG, each to each, and the angle BAC is equal to the angle EDG; therefore the base BC is equal to the base EG (4. 1.); and because DG is equal to DF, the angle DFG is equal to the angle DGF (5. 1.); but the angle DGF is greater than the angle EGF; therefore the angle DFG is greater than EGF; and much more is the angle EFG greater than the angle EGF: and because the angle EFG of the triangle EFG is greater than its angle EGF, and that the greater angle is opposite to the greater side (19. 1.); the side EG is therefore greater than the side EF; but EG is equal to BC; and therefore also BC is greater than EF. Therefore, if two triangles, &c. Q. E. D.

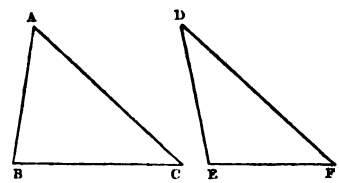


PROP. XXV. THEOR.

If two triangles have two sides of the one equal to two sides of the other, each to each, but the base of the one greater than the base of the other; the angle also, contained by the sides of that which has the greater base shall be greater than the angle contained by the sides equal to them, of the other.

Let ABC, DEF be two triangles which have the two sides AB, AC equal to the two sides DE, DF, each to each, viz. AB equal to DE, and AC to DF; but the base CB greater than the base EF; the angle BAC is likewise greater than the angle EDF.

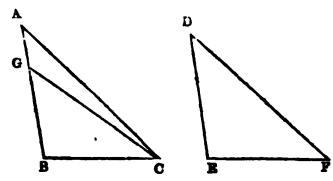
For, if it be not greater, it must either be equal to it, or less; but the angle BAC is not equal to the angle EDF, because then the base BC would be equal (4. 1.) to EF: but it is not; therefore the angle BAC is not equal to the angle EDF: neither is it less; because then the base BC would be less (24. 1.) than the base EF: but it is not; therefore the angle BAC is not less than the angle EDF; and it was shown that it is not equal to it; therefore the angle BAC is greater than the angle EDF. Wherefore if two angles, &c. Q. E. D.



PROP. XXVI. THEOR.

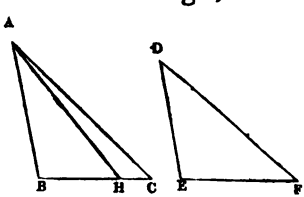
If two triangles have two angles of one equal to two angles of the other, each to each; and one side equal to one side, viz. either the sides adjacent to the equal angles, or the sides opposite to equal angles in each; then shall the other sides be equal, each to each; and also the third angle of the one to the third angle of the other.

Let ABC, DEF be two triangles which have the angles ABC, BCA equal to the angles DEF, EFD, each to each, viz. ABC to DEF; and BCA to EFD; also one side equal to one side; and first let those sides be equal which are adjacent to the angles that are equal in the two triangles, viz. BC to EF; the other sides shall be equal, each to each, viz. AB to DE, and AC to DF, and the third angle, BAC to the third angle, EDF.



For, if AB be not equal to DE, one of them must be the greater. Let AB be the greater of the two, and make BG equal to DE, and join G, C; therefore, because BG is equal to DE, and BC to EF, the two sides GB, BC are equal to the two DE, EF, each to each; and the angle GBC is equal to the angle DEF; therefore the base GC is equal (4. 1.) to the base DF, and the triangle GBC to the triangle DEF, and the other angles to the other angles, each to each, to which the equal sides are opposite; therefore the angle GCB is equal to the angle DFE: but DFE is, by the hypothesis, equal to the angle BCA; wherefore also the angle BCG is equal to the angle BCA, the less to the greater, which is impossible; therefore AB is not unequal to DE, that is, it is equal to it: and BC is equal to EF; therefore the two AB, BC are equal to the two DE, EF, each to each; and the angle ABC is equal to the angle DEF; therefore the base AC is equal (4. 1.) to the base DF, and the third angle, BAC to the third angle, EDF.

Next, let the sides which are opposite to equal angles in each triangle be equal to one another, viz. AB to DE; likewise in this case, the other sides shall be equal, AC to DF, and BC to EF; and also the third angle, BAC to the third, EDF.



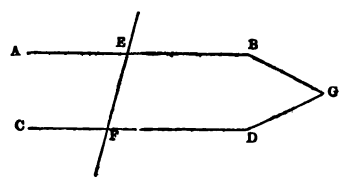
For, if BC be not equal to EF, let BC be the greater of them, and make BH equal to EF, and join A, H; then because BH is equal to EF, and AB to DE; the two AB, BH are equal to the two DE, EF, each to each; and they contain equal angles; therefore the base AH is equal to the base DF, and the triangle ABH to the triangle DEF, and the other angles shall be equal, each to each, to which the equal sides are opposite; therefore the angle BHA is equal to the angle EFD; but EFD is equal to the angle BCA; therefore also the angle BHA is equal to the angle BCA, that is, the exterior angle BHA of the triangle AHC is equal to its interior and opposite angle BCA; which is impossible (16. 1.); wherefore BC is not unequal to EF, that is, it is equal to it; and AB is equal to DE: therefore the two, AB, BC are equal to the two DE, EF, each to each; and they contain equal angles: wherefore the base AC is equal to the base DF, and the third angle BAC to the third angle EDF. Therefore, if two triangles, &c. Q. E. D.

PROP. XXVII. THEOR.

If a straight line falling upon two other straight lines makes the alternate angles equal to one another, these two straight lines shall be parallel.

Let the straight line EF, which falls upon the two straight lines AB, CD, make the alternate angles AEF, EFD equal to one another; AB is parallel to CD.

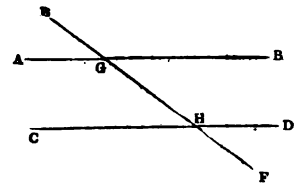
For, if it be not parallel, AB and CD being produced shall meet either towards B, D, or towards A, C; let them be produced and meet towards B, D in the point G; therefore GEF is a triangle, and its exterior angle AEF is greater (16. 1.) than the interior and opposite angle EFG; but it is also equal to it, which is impossible; therefore AB and CD being produced do not meet towards B, D. In like manner it may be demonstrated, that they do not meet towards A, C; but those straight lines which meet neither way, though produced ever so far, are parallel (35. Def.) to one another: AB therefore is parallel to CD. Wherefore, if a straight line, &c. Q. E. D.



PROP. XXVIII. THEOR.

If a straight line falling upon two other straight lines makes the exterior angle equal to the interior and opposite angle upon the same side of the line; or makes the interior angles upon the same side together equal to two right angles; the two straight lines shall be parallel to one another.

Let the straight line EF, which falls upon the two straight lines AB, CD, make the exterior angle EGB equal to the interior and opposite angle GHD upon the same side; or make the interior angles on the same side, BGH, GHD together equal to two right angles; AB is parallel to CD.



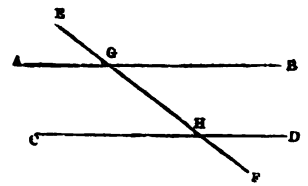
Because the angle EGB is equal to the angle GHD, and the angle EGB equal (15. 1.) to the angle AGH, the angle AGH is equal to the angle GHD; and they are the alternate angles; therefore AB is parallel (27. 1.) to CD. Again because the angles BGH, GHD are equal (by Hyp.) to two right angles; and that AGH, BGH are also equal (13. 1.) to two right angles; the angles AGH, BGH are equal to the angles BGH, GHD: Take away the common angle BGH; therefore the remaining angle AGH is equal to the remaining angle GHD; and they are alternate angles; therefore AB is parallel to CD. Wherefore if a straight line, &c. Q. E. D.

PROP. XXIX. THEOR.

If a straight line fall upon two parallel straight lines it makes the alternate angles equal to one another; and the exterior angle equal to the interior and opposite, upon the same side; and likewise the two interior angles upon the same side together equal to two right angles.

Let the straight line EF fall upon the parallel straight lines AB, CD; the alternate angles AGH, GHD are equal to one another; and the exterior angle EGB is equal to the interior and opposite, upon the same side, GHD; and

the two interior angles BGH, GHD upon the same side are together equal to two right angles.



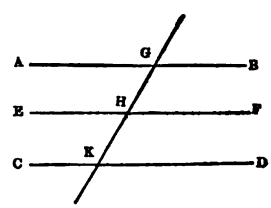
For, if AGH be not equal to GHD, one of them must be greater than the other; let AGH be the greater; and because the angle AGH is greater than the angle GHD, add to each of them the angle BGH; therefore the angles AGH, BGH are greater than the angles BGH, GHD; but the angles AGH, BGH are equal (13. 1.) to two right angles; therefore the angles BGH, GHD are less than two right angles; but those straight lines which, with another straight line falling upon them, make the interior angles on the same side less than two right angles, do meet (5. Post.) together if continually produced; therefore the straight lines AB, CD, if produced far enough, shall meet: but they never meet, since they are parallel by the hypothesis; therefore the angle AGH is not unequal to the angle GHD, that is, it is equal to it; but the angle AGH is equal (15. 1.) to the angle EGB: therefore likewise EGB is equal to GHD; add to each of these the angle BGH: therefore the angles EGB, BGH are equal to the angles BGH, GHD: but EGB, BGH are equal (13. 1.) to two right angles; therefore also BGH, GHD are equal to two right angles. Wherefore, if a straight line, &c. Q. E. D.

PROP. XXX. THEOR.

STRAIGHT lines which are parallel to the same straight line are parallel to each other.

Let AB, CD be each of them parallel to EF; AB is also parallel to CD.

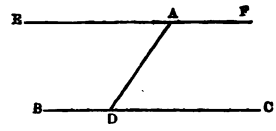
Let the straight line GHK cut AB, EF, CD; and because GHK cuts the parallel straight lines AB, EF, the angle AGH is equal (29. 1.) to the angle GHF. Again, because the straight line GK cuts the parallel straight lines EF, CD, the angle GHF is equal (29. 1.) to the angle GKD: and it was shown that the angle AGK is equal to the angle GHF; therefore also AGK is equal to GKD; and they are alternate angles; therefore AB is parallel (27. 1.) to CD Wherefore straight lines, &c. Q. E. D.



PROP. XXXI. PROB.

To draw a straight line through a given point parallel to a given straight line.

Let A be the given point, and BC the given straight line; it is required to draw a straight line through the point A parallel to the straight line BC.



In BC take any point D, and join A, D; and at the point A, in the straight line AD, make (23. 1.) the angle DAE equal to the angle ADC; and produce the straight line EA to F.

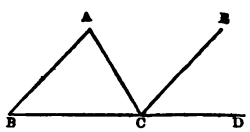
Because the straight line AD, which meets the two straight lines BC, EF, makes the alternate angles EAD, ADC equal to one another, EF is parallel (27. 1.) to BC. Therefore the straight line EAF is drawn through the given point A parallel to the given straight line BC. Q. E. F.

PROP. XXXII. THEOR.

If a side of any triangle be produced, the exterior angle is equal to the two interior and opposite angles; and the three interior angles of every triangle are equal to two right angles.

Let ABC be a triangle, and let one of its sides, BC be produced to D; the exterior angle ACD is equal to the two interior and opposite angles CAB, ABC, and the three interior angles of the triangle, viz. ABC, BCA, CAB, are together equal to two right angles.

Through the point C draw CE parallel (31. 1.) to the straight line AB: and because AB is parallel to CE, and AC meets them, the alternate angles BAC, ACE are equal (29. 1.). Again, because AB is parallel to CE, and BD falls upon them, the exterior angle ECD is equal to the interior and opposite angle ABC; but the angle ACE was shown to be equal to the angle BAC; therefore the whole exterior angle ACD is equal to the two interior and opposite angles CAB, ABC: to these equals add the angle ACB; then the angles ACD, ACB will be equal to the three angles CBA,

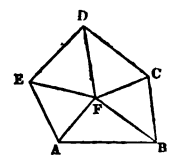


BAC, ACB; but the angles ACD, ACB are equal (13. 1.) to two right angles: therefore also the angles CBA, BAC, ACB are equal to two right angles. Wherefore, if a side of a triangle, &c. Q. E. D.

COR. A. If one angle of a triangle be equal to the sum of the two others, it is a right angle: for the adjacent exterior angle is equal to the sum of the same two angles, by the first part of this proposition; and when the adjacent angles are equal to one another, each of them is a right angle (10. Def.).

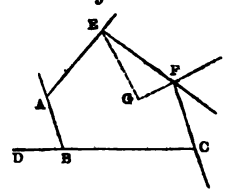
COR. 1. All the interior angles of any rectilineal figure, together with four right angles, are equal to twice as many right angles as the figure has sides.

For any rectilineal figure ABCDE can be divided into as many triangles as the figure has sides, by drawing straight lines from a point F within the figure to each of its angles. And, by the present proposition, all the angles of these triangles are equal to twice as many right angles as there are triangles, that is, as there are sides of the figure; and the same angles are equal to the angles of the figure, together with the angles at the point F, which is the common vertex of the triangles; that is (2. Cor. 13. 1.), together with four right angles. Therefore all the angles of the figure, together with four right angles, are equal to twice as many right angles as the figure has sides.



COR. 2. All the exterior angles of any rectilineal figure are together equal to four right angles.

Because every interior angle ABC, with its adjacent exterior angle ABD, is equal (13. 1.) to two right angles; therefore all the interior together with all the exterior angles of the figure are equal to twice as many right angles as there are sides of the figure; that is, by the foregoing corollary, they are equal to all the interior angles of the figure, together with four right angles: therefore all the exterior angles are equal to four right angles.



N.B. The case in which some of the angles, as EGF, of the figure are re-entering is excluded. If the figure, as BAEGFC, have such an angle, it is evident that the exterior angles of the figure BAEFC will be increased by the angles of the triangle EGF; that is, by two right angles. Hence

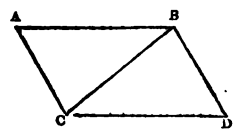
all the exterior angles of any rectilineal figure are equivalent to four right angles together with twice as many right angles as there are re-entering angles.

PROP. XXXIII. THEOR.

THE straight lines which join the extremities of two equal and parallel straight lines, towards the same parts, are also themselves equal and parallel.

Let AB, CD be equal and parallel straight lines, and joined towards the same parts by the straight lines AC, BD; AC, BD are also equal and parallel.

Join B, C; and because AB is parallel to CD, and BC meets them, the alternate angles ABC, BCD are equal (29. 1.); and because AB is equal to CD, and BC common to the two triangles ABC, DCB, the two sides AB, BC are equal to the two DC, CB; and the angle ABC is equal to the angle BCD; therefore the base AC is equal (4. 1.) to the base BD, and the triangle ABC to the triangle BCD, and the other angles to the other angles, each to each, to which the equal sides are opposite; therefore the angle ACB is equal to the angle CBD: and because the straight line BC meets the two straight lines AC, BD, and makes the alternate angles ACB, CBD equal to one another, AC is parallel (27. 1.) to BD; and it was shown to be equal to it. Therefore, straight lines, &c. Q. E. D.



PROP. XXXIV. THEOR.

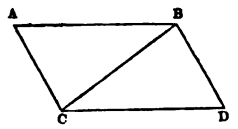
THE opposite sides and angles of a parallelogram are equal to one another, and the diameter, or diagonal, bisects the figure.

N. B. *A parallelogram is a four-sided figure, of which the opposite sides are parallel; and the diameter, or diagonal, is the straight line joining two of its opposite angles.*

Let ACDB be a parallelogram, of which BC is a diameter; the opposite sides and angles of the figure are equal to one another; and the diameter BC bisects it.

Because AB is parallel to CD, and BC meets them, the alternate angles ABC, BCD are equal (29. 1.) to one another; and because AC is parallel to BD, and BC meets them, the

alternate angles ACB, CBD are equal (29. 1.) to one another; wherefore the two triangles ABC, CBD have two angles ABC, BCA in one, equal to two angles BCD, CBD in the other, each to each,



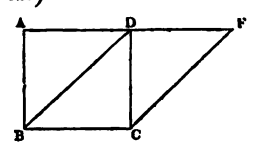
and one side BC common to the two triangles, which is adjacent to their equal angles, therefore their other sides shall be equal, each to each, and the third angle of the one to the third angle of the other (26. 1.), viz. the side AB to the side CD, and AC to BD, and the angle BAC equal to the angle BDC: And because the angle ABC is equal to the angle BCD, and the angle CBD to the angle ACB, the whole angle ABD is equal to the whole angle ACD: And the angle BAC has been shown to be equal to the angle BDC; therefore the opposite sides and angles of parallelograms are equal to one another; also their diameter bisects them; for AB being equal to CD, and BC common, the two AB, BC are equal to the two DC, CB, each to each; and the angle ABC is equal to the angle BCD; therefore the triangle ABC is equal (4. 1.) to the triangle BCD, and the diameter BC divides the parallelogram ACDB into two equal parts. Q. E. D.

PROP. XXXV. THEOR.

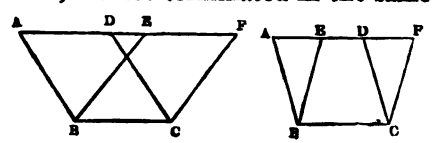
PARALLELOGRAMS upon the same base, and between the same parallels, are equal or equivalent to one another.

Let the parallelograms ABCD, EBCF be upon the same base BC, and between the same parallels AF, BC; the parallelogram ABCD shall be equal or equivalent to the parallelogram EBCF. (See the 2d and 3d figures.)

If the sides AD, DF of the parallelograms ABCD, DBCF, opposite to the base BC, be terminated in the same point D; it is plain that each of the parallelograms is double (34. 1.) of the triangle BDC; and they are therefore equivalent to one another.



But, if the sides AD, EF, opposite to the base BC of the parallelograms ABCD, EBCF, be not terminated in the same point; then, because ABCD is a parallelogram, AD is equal (34. 1.) to BC; for the same reason EF is equal to BC;

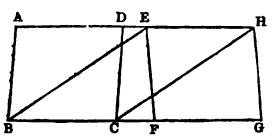


wherefore AD is equal (1. Ax.) to EF; and DE is common; therefore the whole, or the remainder, AE is equal (2. or 3. Ax.) to the whole, or the remainder, DF: AB also is equal to DC; and the two EA, AB are therefore equal to the two FD, DC, each to each; and the exterior angle FDC is equal (29. 1.) to the interior EAB; therefore the base EB is equal to the base FC, and the triangle EAB equal (4. 1.) to the triangle FDC: take the triangle FDC from the trapezium ABCF, and from the same trapezium take the triangle EAB: the remainders therefore are equivalent (3. Ax.); that is, the parallelogram ABCD is equal or equivalent to the parallelogram EBCF. Therefore parallelograms upon the same base, &c. Q. E. D.

PROP. XXXVI. THEOR.

PARALLELOGRAMS upon equal bases, and between the same parallels, are equal or equivalent to one another.

Let ABCD, EFGH be parallelograms upon equal bases BC, FG, and between the same parallels AH, BG: the parallelogram ABCD is equal or equivalent to EFGH.



Join B, E and C, H; and because BC is equal to FG, and FG to (34. 1.) EH, BC is equal to EH; and they are parallels, and joined towards the same parts by the straight lines BE, CH: but straight lines which join equal and parallel straight lines towards the same parts, are themselves equal and parallel (33. 1.); therefore EB, CH are both equal and parallel, and EBCH is a parallelogram; and it is equivalent (35. 1.) to ABCD, because it is upon the same base BC, and between the same parallels BC, AD: for the like reason, the parallelogram EFGH is equivalent to the same EBCH: therefore, also, the parallelogram ABCD is equal or equivalent to EFGH. Wherefore parallelograms, &c. Q. E. D.

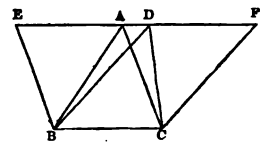
PROP. XXXVII. THEOR.

TRIANGLES upon the same base, and between the same parallels, are equivalent to one another.

Let the triangles ABC, DBC be upon the same base BC,

and between the same parallels AD, BC: the triangle ABC is equivalent to the triangle DBC.

Produce AD both ways to the points E, F, and through B draw (31. 1.) BE parallel to CA; and through C draw CF parallel to BD: therefore each of the figures EBCA, DBCF is a parallelogram; and EBCA is equivalent (35. 1.) to DBCF, because they are upon the same base BC, and between the same parallels BC, EF; and the triangle ABC is the half of the parallelogram, EBCA, because the diameter AB bisects (34. 1.) it; and the triangle DBC is the half of the parallelogram DBCF, because the diameter DC bisects it: but the halves of equal things are equal (7. Ax.); therefore the triangle ABC is equivalent to the triangle DBC. Wherefore triangles, &c. Q. E. D.

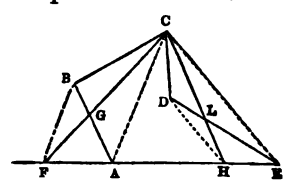


PROP. A. PROB.

To describe a triangle which shall be equivalent to a given rectilineal figure.

Let ABCDE be the given rectilineal figure: it is required to describe a triangle which shall be equivalent to it.

Produce if necessary a side, as EA, of the figure towards F: join C, A; and from B draw BF parallel to AC; also draw CF intersecting BA in G. Then, because the triangles ABC and AFC stand upon the same base AC, and are between the same parallels AC and BF, they are equivalent to one another (37. 1.): take away the common part AGC, and the remainder GBC will be equivalent to the remainder AFG (3. Ax.): add to each of these the figure AGCDE; then the given figure ABCDE will be equivalent to the figure FCDE (2. Ax.), and the latter will have one side less than the former figure.



Again: join C, E; and from D draw DH parallel to CE; also draw CH intersecting DE in L. Then, as before, the triangle CDE will be equivalent to the triangle CHE; and if the common part CLE be taken away, there will remain the triangle CDL equivalent to the triangle LHE: add to each of these the figure FCDLH; then the triangle FCH will be equivalent to the figure FCDE; that is, to the given figure ABCDE. Q. E. F.

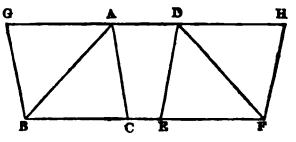
In like manner, whatever be the number of sides which the given figure may have, it may be transformed into a triangle which shall be equivalent to it.

PROP. XXXVIII. THEOR.

TRIANGLES upon equal bases, and between the same parallels, are equal or equivalent to one another.

Let the triangles ABC, DEF be upon equal bases BC, EF, and between the same parallels BF, AD: the triangle ABC, is equal or equivalent to the triangle DEF.

Produce AD both ways to the points G, H, and through B draw BG parallel (31. 1.) to CA, and through F draw FH parallel to ED: then each of the figures GBCA, DEFH is a parallelogram; and they are equivalent to (36. 1.) one another, because they are upon equal bases BC, EF, and between the same parallels BF, GH; and the triangle ABC is the half (34. 1.) of the parallelogram GBCA, because the diameter AB bisects it; and the triangle DEF is the half (34. 1.) of the parallelogram DEFH, because the diameter DF bisects it: but the halves of equal things are equal (7. Ax.); therefore the triangle ABC is equal or equivalent to the triangle DEF. Wherefore triangles, &c. Q. E. D.

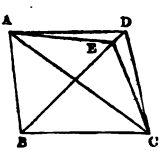


PROP. XXXIX. THEOR.

EQUIVALENT triangles upon the same base, and upon the same side of it, are between the same parallels.

Let the equivalent triangles ABC, DBC be upon the same base BC, and upon the same side of it: they are between the same parallels.

Join A, D; AD is parallel to BC; for, if it is not, through the point A draw (31. 1.) AE parallel to BC, and join E, C: the triangle ABC is equivalent (37. 1.) to the triangle EBC, because it is upon the same base BC, and between the same parallels BC, AE: but the triangle ABC is equivalent to the triangle BDC; therefore also the triangle BDC is equivalent to the triangle EBC, the greater to the less, which is impossible: therefore AE is not parallel to BC. In the same manner it can be demonstrated

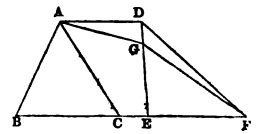


that no other line but AD is parallel to BC; AD is, therefore, parallel to it. Wherefore equivalent triangles upon, &c. Q. E. D.

PROP. XL. THEOR.

EQUAL or equivalent triangles upon equal bases, in the same straight line, and towards the same parts, are between the same parallels.

Let the equal or equivalent triangles ABC, DEF be upon equal bases BC, EF, in the same straight line BF, and towards the same parts: they are between the same parallels.

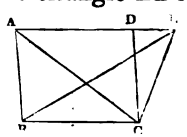


Join A, D; AD is parallel to BC: for, if it is not, through A (31. 1.) draw AG parallel to BF, and join G, F: the triangle ABC is equal or equivalent (38. 1.) to the triangle GEF, because they are upon equal bases BC, EF, and between the same parallels BF, AG: but the triangle ABC is equivalent to the triangle DEF: therefore also the triangle DEF is equivalent to the triangle GEF, the greater to the less, which is impossible: therefore AG is not parallel to BF. And in the same manner it can be demonstrated that there is no other parallel to it but AD; AD is, therefore, parallel to BF. Wherefore equal triangles, &c. Q. E. D.

PROP. XLI. THEOR.

IF a parallelogram and triangle be upon the same base, and between the same parallels: the parallelogram shall be double of the triangle.

Let the parallelogram ABCD and the triangle EBC be upon the same base BC, and between the same parallels BC, AE: the parallelogram ABCD is double of the triangle EBC.



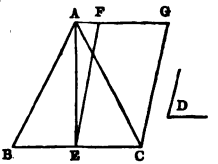
Join A, C: then the triangle ABC is equivalent (37. 1.) to the triangle EBC, because they are upon the same base BC, and between the same parallels BC, AE: but the parallelogram ABCD is double (34. 1.) of the triangle ABC, because the diameter AC divides it into two equal parts; wherefore ABCD is also double of the triangle EBC. Therefore, if a parallelogram, &c. Q. E. D.

PROP. XLII. PROB.

To describe a parallelogram that shall be equivalent to a given triangle, and have one of its angles equal to a given rectilineal angle.

Let ABC be the given triangle, and D the given rectilineal angle: it is required to describe a parallelogram that shall be equivalent to the given triangle ABC, and have one of its angles equal to D.

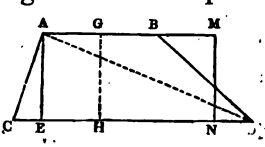
Bisect (10. 1.) BC in E, join A, E, and at the point E in the straight line EC make (23. 1.) the angle CEF equal to D; and through A draw (31. 1.) AG parallel to EC, and through C draw CG (31. 1.) parallel to EF; therefore FECG is a parallelogram: and because BE is equal to EC, the triangle ABE is likewise equivalent (38. 1.) to the triangle AEC, since they are upon equal bases BE, EC, and between the same parallels BC, AG; therefore the triangle ABC is double of the triangle AEC: and the parallelogram FECG is likewise double (41. 1.) of the triangle AEC, because it is upon the same base, and between the same parallels: therefore the parallelogram FECG is equivalent to the triangle ABC, and it has one of its angles, CEF equal to the given angle D; wherefore there has been described a parallelogram FECG equivalent to a given triangle ABC, having one of its angles, CEF equal to the given angle D. Q. E. F.



PROP. B. THEOR.

A TRAPEZOID is equivalent to a right-angled parallelogram between the same parallels, and having a base equal to half the sum of the two parallel sides.

Let ABCD be the trapezoid of which AB and CD are the parallel sides. From A draw AE (11. 1.) perpendicular to AB or CD; then if AM be made equal to half the sum of the sides AB, CD, and a rectangular parallelogram be formed with AM and AE as sides, that parallelogram shall be equivalent to ABDC. Join A, D; bisect AB in G (10. 1.), and on AB, produced if necessary, make GM equal to the half of CD; then will AM be equal to half the sum of AB and CD: draw

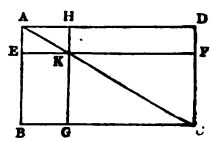


GH and MN parallel to the line AE (31. 1.); then AH, GN and AN will (34. 1.) be right-angled parallelograms: but AH is equivalent to the triangle ABD (42. 1.), and GN is equivalent to the triangle ACD by the same proposition: therefore AN is equivalent to the sum of the two triangles; that is, to the trapezoid ABCD. Q. E. D.

PROP. XLIII. THEOR.

THE complements of the parallelograms, which are about the diameter of any parallelogram, are equal or equivalent to one another.

Let ABCD be a parallelogram, of which the diameter is AC; and EH, FG, the parallelograms about AC, *that is, through which* AC *passes*, and BK, KD, the other parallelograms which make up the whole figure ABCD, which are therefore called the complements: the complement BK is equal or equivalent to the complement KD.



Because ABCD is a parallelogram, and AC its diameter, the triangle ABC is equal (34. 1.) to the triangle ADC: and, because EKHA is a parallelogram, the diameter of which is AK, the triangle AEK is equal to the triangle AHK. By the same reason, the triangle KGC is equal to the triangle KFC: then, because the triangle AEK is equal to the triangle AHK, and the triangle KGC to KFC; the triangle AEK, together with the triangle KGC, is equal to the triangle AHK together with the triangle KFC: but, the whole triangle ABC is equal to the whole ADC; therefore the remaining complement BK is equal or equivalent to the remaining complement KD (3. Ax.). Wherefore the complements, &c. Q. E. D.

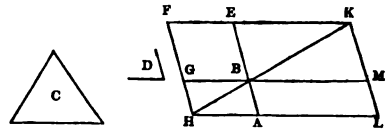
PROP. XLIV. PROB.

To a given straight line to apply a parallelogram, which shall be equivalent to a given triangle, and have one of its angles equal to a given rectilineal angle.

Let AB be the given straight line, and C the given triangle, and D the given rectilineal angle: it is required to apply to the straight line AB a parallelogram equivalent to the triangle C, and having an angle equal to D.

Make (42. 1.) the parallelogram BEFG equivalent to the triangle C, and having the angle EBG equal to the angle D,

so that BE be in the same straight line with AB, and produce FG to H; and through A draw (31. 1.) AH parallel to BG or EF, and join H, B. Then, because the straight line HF falls upon the parallels AH, EF, the angles AHF, HFE are together equal (29. 1.) to two right angles; wherefore the angles BHF, HFE are less than two right angles: but straight lines which, with another straight line, make the interior angles upon the same side less than two right angles, do meet (5. Post.) if produced far enough; therefore HB, FE shall meet if produced; let them meet in K, and through K draw KL, parallel to EA or FH, and produce HA, GB to the points L, M: then HLKF is a parallelogram, of which the diameter is HK, and AG, ME are the parallelograms about HK; and LB, BF are the complements: therefore LB is equivalent (43. 1.) to BF; but BF is equivalent to the triangle C; wherefore LB is equivalent to the triangle C: and because the angle GBE is equal (15. 1.) to the angle ABM, and likewise to the angle D; the angle ABM is equal to the angle D: therefore the parallelogram LB is applied to the straight line AB, is equivalent to the triangle C, and has the angle ABM equal to the angle D. Q. E. F.

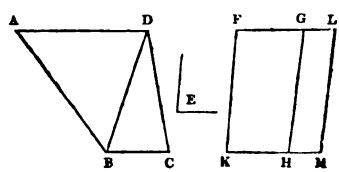


PROP. XLV. PROB.

To describe a parallelogram equivalent to a given rectilineal figure, and having an angle equal to a given rectilineal angle.

Let ABCD be the given rectilineal figure, and E the given rectilineal angle: it is required to describe a parallelogram equivalent to ABCD, and having an angle equal to E.

Join D, B, and describe (42. 1.) the parallelogram FH equivalent to the triangle ADB, having the angle HKF equal to the angle E: and to the straight line GH apply (44. 1.) the parallelogram GM equivalent to the triangle DBC, having the angle GHM equal to the angle E. Then because the angle E is equal to each of the angles FKH, GHM, these last angles are equal to one another: add to each the angle KHG; then since FKH, KHG are equal (29. 1.) to two right angles, KHG, GHM are together



equal to two right angles; therefore (14. 1.) KH is in the same straight line as HM; and because HG meets the parallels KM, FG, the alternate angles MHG, HGF are equal (29. 1.): add to each of these the angle HGL; then HGF, HGL will together be equal to MHG, HGL; that is, to two right angles; therefore FG will be in the same straight line as GL. And because KF is parallel to HG, and HG to ML; KF is parallel (30. 1.) to ML; and KM, FL are parallels; wherefore KFLM is a parallelogram. Now, because the triangle ABD is equivalent to the parallelogram HF, and the triangle DBC to the parallelogram GM; the whole rectilineal figure ABCD is equivalent to the whole parallelogram KFLM; therefore the parallelogram KFLM has been described equivalent to the given rectilineal figure ABCD, &c. Q. E. F.

COR. From this it is manifest how to a given straight line to apply a parallelogram, which shall have an angle equal to a given rectilineal angle, and shall be equivalent to a given rectilineal figure, viz. by applying (44. 1.) to the given straight line a parallelogram equivalent to the first triangle ABD, and having an angle equal to the given angle; and then proceeding as above.

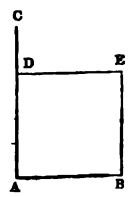
Practically. The given figure may, by prop. B, be transformed into a triangle, which shall be equivalent to it; and then, by prop. 42. or prop. 44. a parallelogram may be described which shall be equivalent to the triangle, and consequently to the given rectilineal figure.

PROP. XLVI. PROB.

To describe a square upon a given straight line.

Let AB be the given straight line: it is required to describe a square upon AB.

From the point A draw (11. 1.) AC at right angles to AB; and make (3. 1.) AD equal to AB, and through the point D draw DE parallel (31. 1.) to AB, and through B draw BE parallel to AD; therefore ADEB is a parallelogram: whence AB is equal (34. 1.) to DE, and AD to BE: but BA is equal to AD; therefore the four straight lines BA, AD, DE, EB are equal to one another, and the parallelogram ADEB is equilateral: likewise all its angles are right angles; because the straight line AD meeting the parallels AB, DE, the angles BAD, ADE are equal (29. 1.) to two right angles. But BAD is a right angle;



therefore also ADE is a right angle: but the opposite angles of parallelograms are equal (34.1.); therefore each of the opposite angles ABE, BED is a right angle; wherefore the figure ADEB is rectangular, and it has been demonstrated that it is equilateral; it is therefore a square, and it is described upon the given straight line AB. Q. E. F.

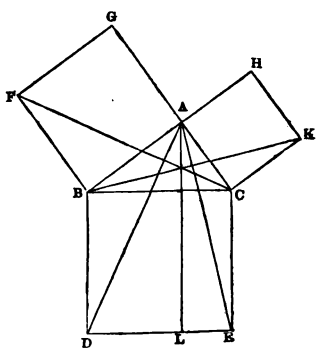
Cor. Hence every parallelogram that has one right angle, has all its angles right angles.

PROP. XLVII. THEOR.

In any right-angled triangle, the square which is described upon the side subtending the right angle, is equivalent to the squares described upon the sides which contain the right angle.

Let ABC be a right-angled triangle having the right angle BAC: the square described upon the side BC is equivalent to the squares described upon BA, AC.

On BC describe (46. 1.) the square BDEC, and on BA, AC the squares GB, HC, and through A draw (31. 1.) AL parallel to BD, or CE; draw also AD, FC: then, because each of the angles BAC, BAG is a right angle (30. Def.), the two straight lines AC, AG, upon the opposite sides of AB, make with it at the point A the adjacent angles equal to two right angles; therefore CA is in the same straight line (14. 1.) with AG; for the same reason, AB and AH are in the same straight line: and because the angle DBC is equal to the angle FBA, each of them being a right angle, add to each the angle ABC; the whole angle DBA is equal (2. Ax.) to the whole FBC; and because the two sides AB, BD are equal to the two FB, BC, each to each, and the angle DBA equal to the angle FBC; therefore the base AD is equal (4. 1.) to the base FC, and the triangle ABD to the triangle FBC. Now the parallelogram BL is double (41. 1.) of the triangle ABD, because they are upon the same base BD, and between the same parallels BD, AL; and the square GB is double of the triangle FBC, because these also



are upon the same base FB, and between the same parallels FB, GC: but the doubles of equals are equal (6. Ax.) to one another; therefore the parallelogram BL is equivalent to the square GB. And, in the same manner, by drawing AE, BK, it is demonstrated that the parallelogram CL is equivalent to the square HC; therefore the whole square BDEC is equivalent to the two squares GB, HC; and the square BDEC is described upon the straight line BC, and the squares GB, HC upon BA, AC: wherefore the square upon the side BC is equivalent to the squares upon the sides BA, AC. Therefore, in any right-angled triangle, &c. Q. E. D.

N.B. *In a right-angled triangle the side which is opposite the right angle is generally called the hypotenuse.*

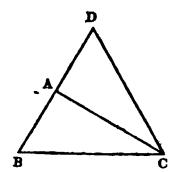
COR. Since, from this proposition, the difference between the squares of the hypotenuse, and of one side, is equal to the square of the other side; it follows that if two right-angled triangles have their hypotenuses equal, or common, and a side of the one equal to a side of the other, the remaining side of the one will be equal to the remaining side of the other. It follows also (8. 1.) that the angles to which the equal sides are opposite are equal to one another; and consequently the triangles are equal in every respect.

PROP. XLVIII. THEOR.

IF the square described upon one of the sides of a triangle, be equivalent to the squares described upon the two other sides of it: the angle contained by these two sides is a right angle.

If the square described upon BC, one of the sides of the triangle ABC, be equivalent to the squares upon the other sides BA, AC, the angle BAC is a right angle.

From the point A draw (11. 1.) AD at right angles to AC, and make AD equal to BA, and join D,C: then, because DA is equal to AB, the square of DA is equal to the square of AB: to each of these add the square of AC; therefore the squares of DA, AC are equal to the squares of BA, AC: but the square of DC is equivalent (47. 1.) to the squares of DA, AC, because DAC is a right angle; and the square of BC, by hypothesis, is equivalent to the squares of BA, AC; therefore the square



of DC is equal to the square of BC; and therefore also the side DC is equal to the side BC: and because the side DA is equal to AB, and AC common to the two triangles DAC, BAC, the two DA, AC are equal to the two BA, AC; and the base DC is equal to the base BC: therefore the angle DAC is equal (8. 1.) to the angle BAC; but DAC is a right angle; therefore also BAC is a right angle. Therefore, if the square, &c. Q. E. D.

ELEMENTS

OF

GEOMETRY.

EUCLID. BOOK II.

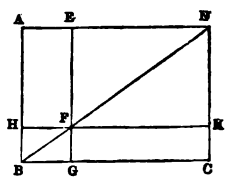
DEFINITIONS.

I.

EVERY right-angled parallelogram is said to be contained by any two of the straight lines which contain one of the right angles.

II.

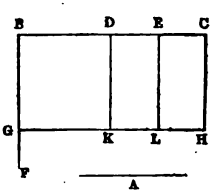
In every parallelogram, any of the parallelograms about a diameter, together with the two complements, is called a Gnomon. 'Thus the parallelogram HG, together with the complements AF, FC, 'is the gnomon, which is more briefly 'expressed by the letters AGK, or 'EHC, at the opposite angles of the 'parallelograms which make the gnomon.'



PROP. I. THEOR.

IF there be two straight lines, one of which is divided into any number of parts: the rectangle contained by the two straight lines, is equal to the rectangles contained by the undivided line, and the several parts of the divided line.

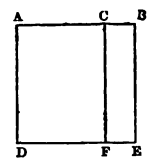
Let A and BC be two straight lines; and let BC be divided into any parts in the points D, E: the rectangle contained by the straight lines A, BC is equal to the rectangle contained



by A, BD, together with that contained by A, DE, and that contained by A, EC. From the point B draw (11. 1.) BF at right angles to BC, and make BG equal (3. 1.) to A; also through G draw (31. 1.) GH parallel to BC; and through D, E, C, draw (31. 1.) DK, EL, CH parallel to BG; then the rectangle BH is equal to the rectangles BK, DL, EH: but BH is contained by A, BC, for it is contained by GB, BC, and GB is equal to A; and BK is contained by A, BD, for it is contained by GB, BD, of which GB is equal to A; and DL is contained by A, DE, because DK, that is (34. 1.) BG is equal to A; and in like manner the rectangle EH is contained by A, EC: therefore the rectangle contained by A, BC is equal to the several rectangles contained by A, BD, and by A, DE, and also by A, EC. Wherefore, if there be two straight lines, &c. Q. E. D.

PROP. II. THEOR.

IF a straight line be divided into any two parts, the rectangles contained by the whole and each of the parts, are together equal to the square of the whole line.



Let the straight line AB be divided into any two parts in the point C; the rectangle contained by AB, BC, together with the rectangle* AB, AC, shall be equal to the square of AB.

Upon AB describe (46. 1.) the square ADEB, and through C draw (31. 1.) CF, parallel to AD or BE: then AE is equal to the rectangles AF, CE; and AE is the square of AB; and AF is the rectangle contained by BA, AC; for it is contained by DA, AC, of which AD is equal to AB; and CE is contained by AB, BC, for BE is equal to AB; therefore the rectangle contained by AB, AC, together with the rectangle AB, BC, is equal to the square of AB. If therefore a straight line, &c. Q. E. D.

PROP. III. THEOR.

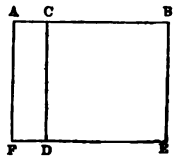
IF a straight line be divided into any two parts, the rectangle contained by the whole and one of the parts

* N. B. To avoid repeating the word *contained* too frequently, the rectangle contained by two straight lines AB, AC is sometimes simply called the rectangle AB, AC.

is equal to the rectangle contained by the two parts, together with the square of the aforesaid part.

Let the straight line AB be divided into two parts in the point C: the rectangle AB, BC is equal to the rectangle AC, CB, together with the square of BC.

Upon BC describe (46. 1.) the square CDEB; produce ED to F, and through A draw (31. 1.) AF parallel to CD or BE; then the rectangle AE is equal to the rectangles AD, CE; and AE is the rectangle contained by AB, BC, for it is contained by AB, BE, of which BE is equal to BC; and AD is contained by AC, CB, for CD is equal to CB; and DB is the square of BC; therefore the rectangle AB, BC, is equal to the rectangle AC, CB, together with the square of BC. If therefore a straight line, &c. Q. E. D.

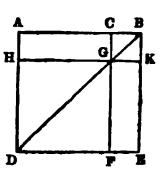


PROP. IV. THEOR.

If a straight line be divided into any two parts, the square of the whole line is equal to the squares of the two parts, together with twice the rectangle contained by the parts.

Let the straight line AB be divided into any two parts in C: the square of AB is equal to the squares of AC, CB, and to twice the rectangle contained by AC, CB.

Upon AB describe (46. 1.) the square ADEB, and join B, D; through C draw (31. 1.) CGF parallel to AD or BE, and through G draw HK parallel to AB or DE. Then because CF is parallel to AD, and BD falls upon them, the exterior angle BGC is equal (29. 1.) to the interior and opposite angle ADB; but ADB is equal (5. 1.) to the angle ABD, because BA is equal to AD, being sides of a square; wherefore the angle CGB is equal to the angle GBC; and therefore the side BC is equal (6. 1.) to the side CG: but CB is equal (34. 1.) also to GK, and CG to BK; wherefore the figure CGKB is equilateral: it is likewise rectangular; for CG is parallel to BK, and CB meets them; the angles KBC, GCB are therefore equal to two right angles; and KBC is a right angle; wherefore GCB is a right angle; and therefore also the angles (34. 1.) CGK, GKB, opposite to these, are right angles, and CGKB



is rectangular; but it is also equilateral, as was demonstrated; wherefore it is a square, and it is upon the side CB. For the same reason HF also is a square, and it is upon the side HG, which is equal to AC: therefore HF, CK are the squares of AC, CB; and because the complement AG is equal (43. 1.) to the complement GE, and that AG is the rectangle contained by AC, CB, for GC is equal to CB; therefore GE is also equal to the rectangle AC, CB; wherefore AG, GE are equal to twice the rectangle AC, CB: and HF, CK are the squares of AC, CB; wherefore the four figures HF, CK, AG, GE are equal to the squares of AC, CB, and to twice the rectangle AC, CB. But HF, CK, AG, GE make up the whole figure ADEB, which is the square of AB: therefore the square of AB is equal to the squares of AC, CB, and twice the rectangle AC, CB. Wherefore, if a straight line, &c. Q. E. D.

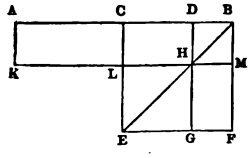
COR. From the demonstration it is manifest that the parallelograms about the diameter of a square are likewise squares.

PROP. V. THEOR.

IF a straight line be divided into two equal parts, and also into two unequal parts, the rectangle contained by the unequal parts, together with the square of the line between the points of section, is equivalent to the square of half the line.

Let the straight line AB be divided into two equal parts in the point C, and into two unequal parts at the point D; the rectangle AD, DB, together with the square of CD, is equivalent to the square of CB.

Upon CB describe (46. 1.) the square CEFB; join B, E; and through D draw (31. 1.) DHG parallel to CE, or BF; and through H draw KLM parallel to CB, or EF; and also through A draw AK parallel to CL, or BM. Then because the complement CH is equal (43. 1.) to the complement HF, to each of these add DM; therefore the whole CM is equal to the whole DF; but CM is equal (36. 1.) to AL, because CB is equal to AC; therefore also AL is equal to DF: to each of these add CH, and the whole AH is equal to DF and CH: but AH is the rectangle contained by AD, DB, for DH is equal (Cor. 4. 2.) to DB; and DF together



with CH is the gnomon CMG; therefore the gnomon CMG is equivalent to the rectangle AD, DB: to each of these add LG, which is equal (Cor. 4. 2.) to the square of CD; therefore the gnomon CMG, together with LG, is equivalent to the rectangle AD, DB, together with the square of CD: but the gnomon CMG and LG make up the whole figure CEFB, which is the square of CB: therefore the rectangle AD, DB, together with the square of CD, is equivalent to the square of CB. Wherefore, if a straight line, &c. Q. E. D.

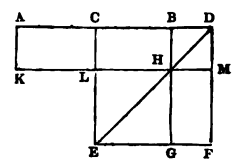
From this proposition it is manifest, that the difference of the squares of two unequal lines AC, CD is equivalent to the rectangle contained by their sum and difference.

PROP. VI. THEOR.

If a straight line be bisected, and produced to any point: the rectangle contained by the whole line thus produced, and the part of it produced, together with the square of half the line bisected, is equivalent to the square of the straight line which is made up of the half and the part produced.

Let the straight line AB be bisected in C, and produced to the point D: the rectangle AD, DB, together with the square of CB, is equivalent to the square of CD.

Upon CD describe (46. 1.) the square CEFD; join D, E; and through B draw (31. 1.) BHG parallel to CE, or DF, and through H draw KLM parallel to AD, or EF, and also through A draw AK parallel to CL, or DM. Then because AC is equal to CB, the rectangle AL is equal (36. 1.) to CH; but CH is equal (43. 1.) to HF; therefore also AL is equal to HF: to each of these add CM; therefore the whole AM is equivalent to the gnomon CMG: and AM is the rectangle contained by AD, DB, for DM is equal (Cor. 4. 2.) to DB: therefore the gnomon CMG is equivalent to the rectangle AD, DB: add to each of these LG, which is equal to the square of CB, therefore the rectangle AD, DB, together with the square of CB, is equivalent to the gnomon CMG, and the figure LG: but the gnomon CMG and LG make up the whole figure CEFD, which is the square of CD; therefore the rectangle AD, DB, together with the square of CB, is equivalent to the square of CD. Wherefore, if a straight line, &c. Q. E. D.

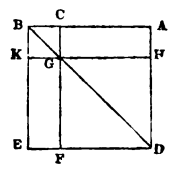


PROP. VII. THEOR.

If a straight line be divided into any two parts: the squares of the whole line, and one of the parts, are equivalent to twice the rectangle contained by the whole and that part, together with the square of the other part.

Let the straight line AB be divided into any two parts in the point C: the squares of AB, BC are equivalent to twice the rectangle AB, BC, together with the square of AC.

Upon AB describe (46. 1.) the square ADEB, and construct the figure as in the preceding propositions: and because AG is equal (43. 1.) to GE, add to each of them CK; the whole AK is therefore equal to the whole CE; therefore AK, CE are double of AK: but AK, CE are equivalent to the gnomon AKF, together with the square CK; therefore the gnomon AKF, together with the square CK, is equivalent to the double of AK: but twice the rectangle AB, BC is double of AK, for BK is equal (Cor. 4. 2.) to BC; therefore the gnomon AKF, together with the square CK, is equivalent to twice the rectangle AB, BC. To each of these equivalents add HF, which is equal to the square of AC; therefore the gnomon AKF, together with the squares CK, HF, is equivalent to twice the rectangle AB, BC, and the square of AC: but the gnomon AKF, together with the squares CK, HF, make up the whole figure ADEB and CK, which are the squares of AB and BC; therefore the squares of AB and BC are equivalent to twice the rectangle AB, BC, together with the square of AC. Wherefore, if a straight line, &c. Q. E. D.



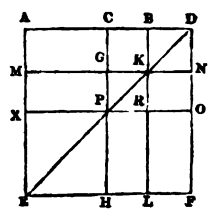
PROP. VIII. THEOR.

If a straight line be divided into any two parts: four times the rectangle contained by the whole line, and one of the parts, together with the square of the other part, is equivalent to the square of the straight line, which is made up of the whole and that part.

Let the straight line AB be divided into any two parts in the point C: four times the rectangle AB, BC, together with

the square of AC, is equivalent to the square of the straight line made up of AB and BC together.

Produce AB to D, so that BD be equal to CB, and upon AD describe the square AEFD; and construct two figures such as in the preceding propositions. Because CB is equal to BD, and that CB is equal (34. 1.) to GK, and BD to KN; therefore GK is equal to KN: for the same reason, PR is equal to RO; and the rectangle CK is equal (36. 1.) to BN, and GR to RN: but CK is equal (43. 1.) to RN, because they are the complements of the parallelogram CO; therefore also BN is equal to GR; and the four rectangles BN, CK, GR, RN are therefore equal to one another: they are, therefore, together quadruple of one of them, CK. Again, because CB is equal to BD, and that BD is equal (Cor. 4. 2.) to BK, that is, to CG, and CB equal to GK, that is (Cor. 4. 2.), to GP; therefore CG is equal to GP: and because CG is equal to GP, and PR to RO, the rectangle AG is equal to MP, and PL to RF. But MP is equal (43. 1.) to PL, because they are the complements of the parallelogram ML; wherefore AG is equal also to RF; therefore the four rectangles AG, MP, PL, RF are equal to one another, and thus are together quadruple of one of them, AG. And it was demonstrated that the four, CK, BN, GR, and RN are quadruple of CK. Therefore the eight rectangles which contain the gnomon AOH, are quadruple of AK; and because AK is the rectangle contained by AB, BC, for BK is equal to BC, four times the rectangle AB, BC is quadruple of AK. But the gnomon AOH was demonstrated to be quadruple of AK; therefore four times the rectangle AB, BC is equal to the gnomon AOH. To each of these add XH, which is equal (Cor. 4. 2.) to the square of AC: therefore four times the rectangle AB, BC, together with the square of AC, is equal to the gnomon AOH and the square XH: but the gnomon AOH and XH make up the figure AEFD, which is the square of AD: therefore four times the rectangle AB, BC, together with the square of AC, is equivalent to the square of AD, that is, of AB and BC added together in one straight line. Wherefore, if a straight line, &c. Q. E. D.



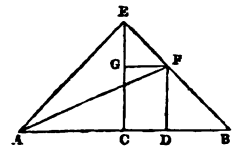
PROP. IX. THEOR.

If a straight line be divided into two equal, and also into two unequal parts: the squares of the two un-

equal parts are together equivalent to double of the square of half the line, and of the square of the line between the points of section.

Let the straight line AB be divided at the point C into two equal, and at D into two unequal parts: the squares of AD, DB are together equivalent to double of the squares of AC, CD.

From the point C draw (11. 1.) CE at right angles to AB, and make it equal to AC or CB, and draw EA, EB; through D draw (31. 1.) DF parallel to CE, and through F draw FG parallel to AB; and join A, F. Then, because AC is equal to CE, the angle EAC is equal (5. 1.) to the angle AEC; and because the angle ACE is a right angle, the two others AEC, EAC together make one right angle (32. 1.); and they are equal to one another; each of them, therefore, is half of a right angle. For the same reason each of the angles CEB, EBC is half a right angle; and therefore the whole AEB is a right angle. And because the angle GEF is half a right angle, and EGF a right angle, for it is equal (29. 1.) to the interior and opposite angle ECB, the remaining angle EFG is half a right angle; therefore the angle GEF is equal to the angle EFG, and the side EG equal (6.1.) to the side GF. Again: because the angle at B is half a right angle, and FDB a right angle, for it is equal (29. 1.) to the interior and opposite angle ECB, the remaining angle BFD is half a right angle; therefore the angle at B is equal to the angle BFD, and the side DF (6. 1.) to the side DB: and because AC is equal to CE, the square of AC is equal to the square of CE; but the square of EA is equivalent (47. 1.) to the squares of AC, CE, because ACE is a right angle; therefore the square of EA is equivalent to double the square of AC. Again: because EG is equal to GF, the square of EF is, for a like reason, equivalent to double the square of GF: but GF is equal (34. 1.) to CD; therefore the square of EF is equivalent to double the square of CD, and the squares of AE, EF are together double the squares of AC, CD. But the square of AF is equivalent (47. 1.) to the squares of AE, EF, because AEF is a right angle; also the square of AF is equivalent (47. 1.) to the squares of AD, DF, because ADF is a right angle; therefore the squares of AD, DF are together equivalent to double the squares of AC, CD. But DF is equal to DB; therefore the squares of AD, DB are equivalent to double the squares of AC, CD. If therefore a straight line, &c. Q. E. D.

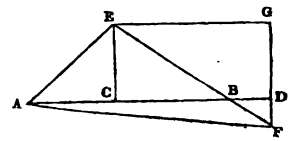


PROP. X. THEOR.

If a straight line be bisected, and produced to any point: the square of the whole line thus produced, and the square of the part of it produced, are together equivalent to double of the square of half the line bisected, and of the square of the line made up of the half and the part produced.

Let the straight line AB be bisected in C and produced to the point D: the squares of AD, DB are equivalent to double of the squares of AC, CD.

From the point C draw (11. 1.) CE at right angles to AB: and make it equal to AC or CB, and draw AE, EB; through E draw (31. 1.) EG parallel to AB, and through D draw DG parallel to CE. And because the straight line EG meets the parallels EC, GD, the angles CEG, EGD are equal (29. 1.) to two right angles:



and therefore the angles BEG, EGD are less than two right angles. But straight lines which with another straight line make the interior angles upon the same side less than two right angles do meet (5. Post.) if produced far enough: therefore EB, GD shall meet, if produced towards B, D. Let them meet in F, and join AF: then, because AC is equal to CE, the angle CEA is equal (5. 1.) to the angle EAC; and the angle ACE is a right angle; therefore each of the angles CEA, EAC is half a right angle (32. 1.). For the same reason, each of the angles CEB, EBC is half a right angle; therefore AEB is a right angle. And because EBC is half a right angle, DBF is also (15. 1.) half a right angle, for they are vertically opposite; but BDF is a right angle, because it is equal (29. 1.) to the alternate angle DCE; therefore the remaining angle DFB is half a right angle, and is therefore equal to the angle DBF; wherefore also the side BD is equal (6. 1.) to the side DF. Again, because EFG is half a right angle, and that the angle at G is a right angle, because it is equal (34. 1.) to the opposite angle ECD, the remaining angle FEG is half a right angle, and equal to the angle EFG; wherefore also the side GF is equal (6. 1.) to the side GE. And because AC is equal to CE, the square of AC is equal to the square of CE; but the square of EA is equivalent (47. 1.) to the squares of AC, CE, because ACE is a right angle; therefore the square of EA is equivalent to double the square

of AC. Again, because EG is equal to GF, the square of EF is, for a like reason, equivalent to double the square of EG: but EG is equal (34. 1.) to CD; therefore the square of EF is equivalent to double the square of CD, and the squares of AE, EF are together double the squares of AC, CD. But the square of AF is equivalent (47. 1.) to the squares of AE, EF, because AEF is a right angle; and also the square of AF is equivalent (47. 1.) to the squares of AD, DF, because ADF is a right angle; therefore the squares of AD, DF are together equivalent to double the squares of AC, CD. Now DF is equal to DB; therefore the squares of AD, DB are equivalent to double the squares of AC, CD. If therefore a straight line, &c. Q. E. D.

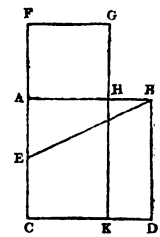
PROP. XI. PROB.

To divide a given straight line into two parts, so that the rectangle contained by the whole, and one of the parts, shall be equivalent to the square of the other part.

Let AB be the given straight line: it is required to divide it into two parts, so that the rectangle contained by the whole, and one of the parts, shall be equivalent to the square of the other part.

Upon AB describe (46. 1.) the square ABDC; bisect (10. 1.) AC in E, and join B, E; produce CA to F, and make (3. 1.) EF equal to EB, and upon AF describe (46. 1.) the square FGHA: AB is divided in H, so that the rectangle AB, BH is equivalent to the square of AH.

Produce GH to K: because the straight line AC is bisected in E, and produced to the point F, the rectangle CF, FA, together with the square of AE, is equivalent (6. 2.) to the square of EF: but EF is equal to EB; therefore the rectangle CF, FA, together with the square of AE, is equivalent to the square of EB; and the squares of BA, AE are equivalent (47. 1.) to the square of EB, because the angle EAB is a right angle; therefore the rectangle CF, FA, together with the square of AE, is equivalent to the squares of BA, AE. Take away the square of AE, which is common to both; therefore the remaining rectangle CF, FA is equivalent to the square of AB: but the figure FK is the rectangle contained by CF, FA, for AF is equal to FG, and AD is the square of AB; therefore FK is equivalent to AD.



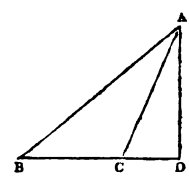
Take away the common part AK, and the remainder FH is equivalent to the remainder HD; and HD is the rectangle contained by AB, BH, for AB is equal to BD; and FH is the square of AH; therefore the rectangle AB, BH is equivalent to the square of AH: wherefore, the straight line AB is divided in H, so that the rectangle AB, BH is equivalent to the square of AH. Q. E. F.

PROP. XII. THEOR.

In obtuse-angled triangles, if a perpendicular be drawn from any of the acute angles to the opposite side produced, the square of the side subtending the obtuse angle is greater than the squares of the sides containing the obtuse angle, by twice the rectangle contained by the side upon which, when produced, the perpendicular falls, and the straight line intercepted on the exterior of the triangle between the perpendicular and the obtuse angle.

Let ABC be an obtuse-angled triangle, having the obtuse angle ACB, and from the point A let AD be drawn (12. 1.) perpendicular to BC produced: the square of AB is greater than the squares of AC, CB, by twice the rectangle BC, CD.

Because the straight line BD is divided into two parts in the point C, the square of BD is equal (4. 2.) to the squares of BC, CD, and twice the rectangle BC, CD. To each of these equals add the square of DA, and the squares of BD, DA are equal to the squares BC, CD, DA, and twice the rectangle BC, CD: but the square of BA is equivalent (47. 1.) to the squares of BD, DA, because the angle at D is a right angle; and the square of CA is equivalent (47. 1.) to the squares of CD, DA: therefore the square of BA is equivalent to the squares of BC, CA, and twice the rectangle BC, CD; that is, the square of BA is greater than the squares of BC, CA, by twice the rectangle BC, CD. Therefore, in obtuse-angled triangles, &c. Q. E. D.



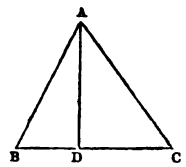
PROP. XIII. THEOR.

In every triangle, the square of the side subtending any of the acute angles, is less than the squares of

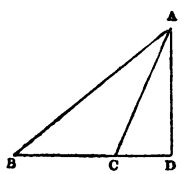
the sides containing that angle, by twice the rectangle contained by either of these sides, and the straight line intercepted between the perpendicular let fall upon it from the opposite angle, and the acute angle.

Let ABC be any triangle, and the angle at B one of its acute angles, and upon BC, one of the sides containing it, let fall the perpendicular (12. 1.) AD from the opposite angle: the square of AC, opposite to the angle B, is less than the squares of CB, BA, by twice the rectangle CB, BD.

First, let AD fall within the triangle ABC: and because the straight line CB is divided into two parts in the point D, the squares of CB, BD are equivalent (7. 2.) to twice the rectangle contained by CB, BD, and the square of DC. To each of these equals add the square of AD; therefore the squares of CB, BD, DA are equivalent to twice the rectangle CB, BD, and the squares of AD, DC: but the square of AB is equivalent (47. 1.) to the squares of BD, DA, because the angle BDA is a right angle; and the square of AC is equivalent to the squares of AD, DC: therefore the squares of CB, BA are equivalent to the square of AC, and twice the rectangle CB, BD, that is, the square of AC alone is less than the squares of CB, BA, by twice the rectangle CB, BD.



Secondly, Let AD fall on the exterior of the triangle ABC: then, because the straight line BD is divided into two parts in the point C, the squares of DB and BC are equal to twice the rectangle DB, BC, and the square of DC (7. 2.). Therefore, to each of these equals adding the square of AD, and reasoning as before, it will follow that the square of AC is less than the squares of CB and BA, by twice the rectangle CB, BD.



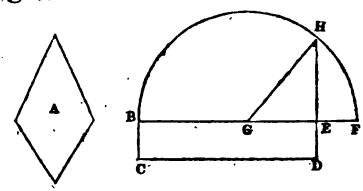
Lastly, Let the side AC be perpendicular to BC; then is BC the straight line between the perpendicular and the acute angle at B; and it is manifest that the squares of AB, BC are equivalent (47. 1.) to the square of AC and twice the square of BC: therefore, in every triangle, &c. Q. E. D.

PROP. XIV. PROB.

To describe a square that shall be equivalent to a given rectilineal figure.

Let A be the given rectilineal figure: it is required to describe a square that shall be equivalent to A.

Describe (45. 1.) the rectangular parallelogram BCDE equivalent to the rectilineal figure, A. If then, the sides of it BE, ED are equal to one another, it is a square, and what was required is now done: but if they are not equal, produce one of them, BE to F, and make EF equal to ED, and bisect BF in G; and from the centre G, at the distance GB, or GF, describe the semicircle BHF, and produce DE to H, and join G, H. Therefore, because the straight line BF is divided into two equal parts in the point G, and into two unequal parts at E, the rectangle BE, EF, together with the square of EG, is equivalent (5. 2.) to the square of GF: But GF is equal to GH; therefore the rectangle BE, EF, together with the square of EG, is equivalent to the square of GH: but the squares of HE, EG are equivalent (47. 1.) to the square of GH: therefore the rectangle BE, EF, together with the square of EG, is equivalent to the squares of HE, EG. Take away the square of EG, which is common to both; and the remaining rectangle BE, EF is equivalent to the square of EH: but the rectangle contained by BE, EF is the parallelogram BD, because EF is equal to ED: therefore BD is equivalent to the square of EH; but BD is equivalent to the rectilineal figure A: therefore the rectilineal figure A is equivalent to the square of EH. Wherefore a square has been made equivalent to the given rectilineal figure A, viz. the square described upon EH. Q. E. F.



ELEMENTS

OF

GEOMETRY.

EUCLID. BOOK III.

DEFINITIONS.

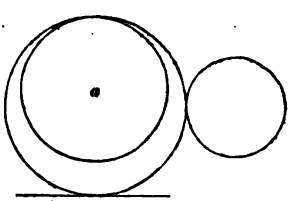
I.

EQUAL circles are those of which the diameters are equal, or from the centres of which the straight lines to the circumferences are equal.

'This is not a definition, but a theorem, the truth of which 'is evident; for, if the circles be applied to one another, so 'that their centres coincide, the circles must likewise coincide, 'since the straight lines from the centres are equal.'

II.

A straight line is said to touch a circle, when it meets the circle, and being produced does not cut it.

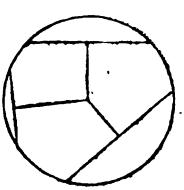


III.

Circles are said to touch one another, which meet but do not cut one another.

IV.

Straight lines are said to be equally distant from the centre of a circle, when the perpendiculars drawn to them from the centre are equal.



V.

And the straight line on which the greater perpendicular falls, is said to be farther from the centre.

VI.

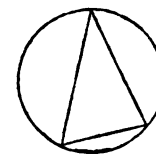
A segment of a circle is the figure contained by a straight line and the circumference it cuts off.

VII.

"The angle of a segment is that which is contained by the "straight line and the circumference."

VIII.

An angle in a segment is the angle contained by two straight lines drawn from any point in the circumference of the segment to the extremities of the straight line which is the base of the segment.

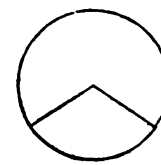


IX.

And an angle is said to insist, or stand, upon the circumference intercepted between the straight lines that contain the angle.

X.

The sector of a circle is the figure contained by two straight lines drawn from the centre, and the circumference between them.



XI.

Similar segments of a circle are those in which the angles are equal, or which contain equal angles.

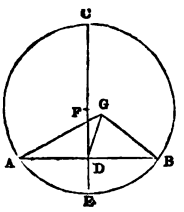
PROP. I. PROB.

To find the centre of a given circle.

Let ABC be the given circle: it is required to find its centre.

Draw within it any straight line AB, and bisect (10. 1.) it in D; from the point D draw (11. 1.) DC at right angles to AB, and produce it to E, and bisect CE in F: the point F is the centre of the circle ABC.

For, if it be not, let, if possible, G be the centre, and draw GA, GD, GB: then, because DA is equal to DB, and

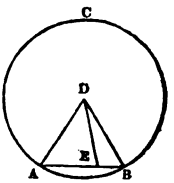


DG common to the two triangles ADG, BDG, the two sides AD, DG are equal to the two BD, DG, each to each; and the base GA is equal to the base GB, because they are drawn from the centre G*: therefore the angle ADG is equal (8. 1.) to the angle GDB. But when a straight line standing upon another straight line makes the adjacent angles equal to one another, each of the angles is a right angle (10. Def. 1.); therefore the angle GDB is a right angle: but FDB is likewise a right angle; wherefore the angle FDB is equal to the angle GDB, the greater to the less, which is impossible: therefore G is not the centre of the circle ABC. In the same manner it can be shown, that no other point but F is the centre; that is, F is the centre of the circle ABC. Which was to be found.

COR. From this it is manifest, that if in a circle a straight line bisect another at right angles, the centre of the circle is in the line which bisects the other.

PROP. II. THEOR.

IF any two points be taken in the circumference of a circle, the straight line which joins them shall fall within the circle.



Let ABC be a circle, and A, B any two points in the circumference: the straight line drawn from A to B shall fall within the circle.

For, find (1. 3.) D the centre of the circle ABC; and draw AD, DB: then, because DA is equal to DB, the angle DAB is equal (5. 1.) to the angle DBA. Take any point E in AB and join D, E: and because AE, a side of the triangle DAE, is produced to B, the angle DEB is greater (16. 1.) than the angle DAE; but DAE is equal to the angle DBE; therefore the angle DEB is greater than the angle DBE: but to the greater angle the greater side is opposite (19. 1.); DB is therefore greater than DE. Now DB is a semi-diameter of the circle; therefore DE is less than a semi-diameter, and the point E is

* N. B. Whenever the expression "straight lines from the centre," or "drawn from the centre," occurs, it is to be understood that they are drawn to the circumference.

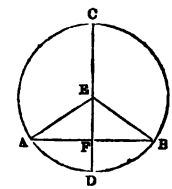
within the circle. In like manner any point in the line AB except A and B may be shown to be within the circle. Wherefore, &c. Q. E. D.

PROP. III. THEOR.

If a straight line drawn through the centre of a circle bisect a straight line in it which does not pass through the centre, it shall cut it at right angles; and if it cut it at right angles, it shall bisect it.

Let ABC be a circle; and let CD, a straight line drawn through the centre, bisect any straight line AB, which does not pass through the centre, in the point F: it cuts it also at right angles.

Take (1. 3.) E, the centre of the circle, and draw EA, EB: then, because AF is equal to FB, and FE common to the two triangles AFE, BFE, there are two sides in the one equal to two sides in the other, and the base EA is equal to the base EB; therefore the angle AFE is equal (8. 1.) to the angle BFE. But when a straight line standing upon another makes the adjacent angles equal to one another, each of them is a right angle (10. Def. 1.): therefore each of the angles AFE, BFE is a right angle; wherefore the straight line CD, drawn through the centre, bisecting another, AB that does not pass through the centre, cuts the same at right angles.



But let CD cut AB at right angles; CD also bisects it, that is, AF is equal to FB.

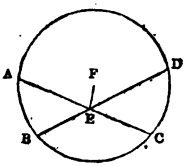
The same construction being made; because EA, EB from the centre are equal to one another, the angle EAF is equal (5. 1.) to the angle EBF; and the right angle AFE is equal to the right angle BFE: therefore, in the two triangles EAF, EBF, there are two angles in one equal to two angles in the other, and the side EF, which is opposite to one of the equal angles in each, is common to both; therefore the other sides are equal (26. 1.); AF therefore is equal to FB. Wherefore, if a straight line, &c. Q. E. D.

PROP. IV. THEOR.

If in a circle two straight lines cut one another, which do not both pass through the centre, they do not bisect each other.

Let ABCD be a circle, and AC, BD two straight lines in it which cut one another in the point E, and do not both pass through the centre: AC, BD do not bisect one another.

For, if it be possible, let AE be equal to EC, and BE to ED: If one of the lines pass through the centre, it is plain that it cannot be bisected by the other which does not pass through the centre: but if neither of them pass through the centre, take (1. 3.) F the centre of the circle, and join E, F; and because FE, a straight line through the centre, bisects another, AC which does not pass through the centre, it shall cut it at right angles (3. 3.); wherefore FEA is a right angle. Again, because the straight line FE bisects the straight line BD, which does not pass through the centre, it shall cut it at right angles; wherefore FEB is a right angle: and FEA was shown to be a right angle; therefore FEA is equal to the angle FEB, the less to the greater, which is impossible: therefore AC, BD do not bisect one another. Wherefore, if in a circle, &c. Q. E. D.

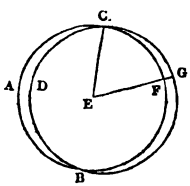


PROP. V. THEOR.

If two circles cut one another, they shall not have the same centre.

Let the two circles ABC, CDG cut one another in the points B, C: they have not the same centre.

For, if it be possible, let E be their centre. Join E, C, and draw any straight line EFG meeting them in F and G; and because E is the centre of the circle ABC, CE is equal to EF. Again, because E is the centre of the circle CDG, CE is equal to EG: but CE was shown to be equal to EF; therefore EF is equal to EG, the less to the greater, which is impossible: therefore E is not the centre of the circles ABC, CDG. Wherefore, if two circles, &c. Q. E. D.

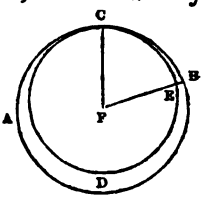


PROP. VI. THEOR.

If two circles touch one another internally, they shall not have the same centre.

Let the two circles ABC, CDE touch one another internally in the point C: they have not the same centre.

For, if they have, let it be F; join F, C, and draw any straight line FEB meeting them in E and B. And because F is the centre of the circle ABC, CF is equal to FB; also, because F is the centre of the circle CDE, CF is equal to FE: and CF was shown to be equal to FB; therefore FE is equal to FB, the less to the greater, which is impossible; wherefore F is not the centre of the circles ABC, CDE. Therefore, if two circles, &c. Q. E. D.

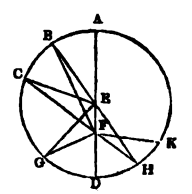


PROP. VII. THEOR.

If any point be taken in the diameter of a circle which is not the centre: of all the straight lines which can be drawn from it to the circumference, the greatest is that in which the centre is, and the other part of that diameter is the least; and of any others, that which is nearer to the line passing through the centre is always greater than one more remote: and from the same point there can be drawn only two straight lines that are equal to one another, one upon each side of the diameter.

Let ABCD be a circle; E its centre, and AD its diameter, in which let any point F be taken which is not the centre: of all the straight lines FB, FC, FG, &c. that can be drawn from F to the circumference, FA is the greatest, and FD, the other part of the diameter AD, is the least: and of the others, FB is greater than FC, and FC than FG.

Draw BE, CE, GE: and because two sides of a triangle are greater (20. 1.) than the third, BE, EF are greater than BF: but AE is equal to EB; therefore AE, EF, that is AF, is greater than BF. Again: because BE is equal to CE, and FE common to the triangles BEF, CEF, the two sides BE, EF are equal to the two CE, EF; but the angle BEF is greater than the angle CEF; therefore the base BF is greater (24. 1.) than the base FC: for the same reason, CF is greater than GF. Again: because GF, FE are greater (20. 1.) than EG, and EG is equal to ED; GF, FE are greater than ED. Take away the common part FE, and the remainder GF



is greater than the remainder FD: therefore FA is the greatest, and FD the least, of all the straight lines from F to the circumference; and BF is greater than CF, and CF than GF.

Also there can be drawn only two equal straight lines from the point F to the circumference, one upon each side of the diameter AD. At the point E, in the straight line EF, make (23. 1.) the angle FEH equal to the angle GEF, and join F, H: then, because GE is equal to EH, and EF common to the two triangles GEF, HEF; the two sides GE, EF are equal to the two HE, EF; and the angle GEF is equal to the angle HEF; therefore the base FG is equal (4. 1.) to the base FH: but, besides FH, no other straight line can be drawn from F to the circumference equal to FG: for, if there can, let it be FK; and because FK is equal to FG, and FG to FH, FK is equal to FH; that is, a line nearer to that which passes through the centre, is equal to one which is more remote; which is impossible. Therefore, if any point be taken, &c. Q.E.D.

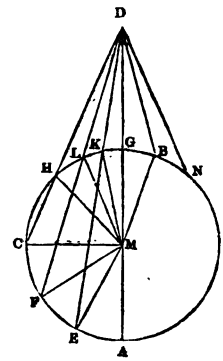
PROP. VIII. THEOR.

If any point be taken on the exterior of a circle, and straight lines be drawn from it to the circumference, whereof one passes through the centre: of those which fall upon the concave circumference, the greatest is that which passes through the centre; and of the rest, that which is nearer to the one through the centre is always greater than one more remote: but of those which fall upon the convex circumference, the least is that between the point on the exterior of the circle and the diameter; and of the rest, that which is nearer to the least is always less than the more remote: and only two equal straight lines can be drawn from the point to the circumference, one upon each side of the least line.

Let ABC be a circle, and D any point on its exterior, from which let the straight lines DA, DE, DF, DC be drawn to the circumference, whereof DA passes through the centre. Of those which fall upon the concave part of the circumference AEFC, the greatest is AD which passes through the centre; and the nearer to it is always greater than the more remote, viz. DE than DF, and DF than DC: but of those which fall upon the convex circumference HLKG, the least is DG, between the point D and the diameter AG; and the nearer to it

is always less than the more remote, viz. DK than DL, and DL than DH.

Take (1. 3.) M the centre of the circle ABC, and draw ME, MF, MC, MK, ML, MH. And because AM is equal to ME, add MD to each, therefore AD is equal to EM, MD; but EM, MD are greater (20. 1.) than ED; therefore also AD is greater than ED. Again, because ME is equal to MF, and MD common to the triangles EMD, FMD; EM, MD are equal to FM, MD; but the angle EMD is greater than the angle FMD; therefore the base ED is greater (24. 1.) than the base FD. In like manner it may be shown that FD is greater than CD: therefore DA is the greatest; and DE greater than DF, and DF than DC. And because MK, KD are greater (20. 1.) than MD, and MK is equal to MG, the remainder KD is greater (5. Ax.) than the remainder GD, that is GD is less than KD. And because MK, DK are drawn to the point K within the triangle MLD from M, D, the extremities of its side MD; MK, KD are less (21. 1.) than ML, LD, whereof MK is equal to ML; therefore the remainder DK is less than the remainder DL. In like manner it may be shown that DL is less than DH: therefore DG is the least, and DK less than DL, and DL than DH. Also there can be drawn only two equal straight lines from the point D to the circumference, one upon each side of the least. At the point M, in the straight line MD, make the angle DMB equal to the angle DMK, and join DB: and because MK is equal to MB, and MD common to the triangles KMD, BMD, the two sides KM, MD are equal to the two BM, MD; and the angle KMD is equal to the angle BMD; therefore the base DK is equal (4. 1.) to the base DB: but besides DB, there can be no straight line drawn from D to the circumference equal to DK: for, if there can, let it be DN; and because DK is equal to DN, and also to DB; therefore DB is equal to DN, that is, the nearer to the least is equal to the more remote, which is impossible. If therefore, any point, &c. Q. E. D.

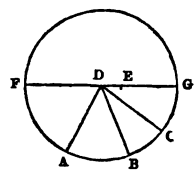


PROP. IX. THEOR.

If a point be taken within a circle, from which there fall more than two equal straight lines to the circumference, that point is the centre of the circle.

Let the point D be taken within the circle ABC, from which to the circumference there fall more than two equal straight lines, viz. DA, DB, DC: the point D is the centre of the circle.

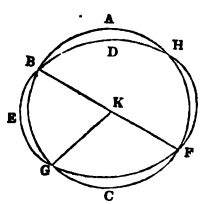
For, if not, let E be the centre; draw DE, and produce it to the circumference in F, G: then FG is a diameter of the circle ABC. And because in FG, the diameter of the circle ABC, there is taken the point D, which is not the centre, DG shall be the greatest line from it to the circumference, and DC greater (7. 3.) than DB, and DB than DA: but they are likewise equal, which is impossible; therefore E is not the centre of the circle ABC. In like manner it may be demonstrated that no other point but D is the centre; D therefore is the centre. Wherefore, if a point be taken, &c. Q. E. D.



PROP. X. THEOR.

ONE circumference of a circle cannot cut another in more than two points.

If it be possible, let the circumference ABC cut the circumference DEF in more than two points, viz. in B, G, F; take the centre K of the circle ABC, and draw KB, KG, KF. And because within the circle DEF there is taken the point K, from which to the circumference DEF fall more than two equal straight lines KB, KG, KF, the point K is (9. 3.) the centre of the circle DEF: but K is also the centre of the circle ABC; therefore the same point is the centre of two circles that cut one another, which is impossible (5. 3.). Therefore one circumference of a circle cannot cut another in more than two points. Q. E. D.

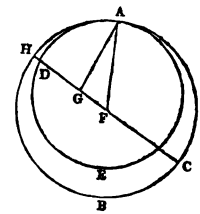


PROP. XI. THEOR.

IF two circles touch each other internally, the straight line which joins their centres being produced shall pass through the point of contact.

Let the two circles ABC, ADE touch each other internally in the point A, and let F be the centre of the circle

ABC, and G the centre of the circle ADE: the straight line which joins the centres F, G, being produced, passes through the point A.



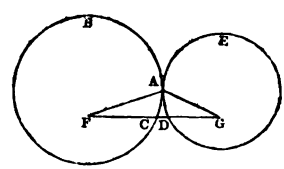
For, if not, let it fall otherwise, if possible, as FGDH, and draw AF, AG. And because AG, GF are greater (20. 1.) than FA, that is, than FH, for FA is equal to FH, both being drawn from the same centre; take away the common part FG; therefore the remainder AG is greater than the remainder GH: but AG is equal to GD; therefore GD is greater than GH, the less than the greater, which is impossible: therefore the straight line which joins the points F, G cannot fall otherwise than upon the point A, that is, it must pass through it. Therefore, if two circles, &c. Q. E. D.

PROP. XII. THEOR.

If two circles touch each other externally, the straight line which joins their centres shall pass through the point of contact.

Let the two circles ABC, ADE touch each other externally in the point A; and let F be the centre of the circle ABC, and G the centre of ADE: the straight line which joins the points F, G shall pass through the point of contact, A.

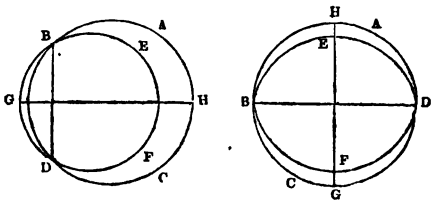
For, if not, let it pass otherwise, if possible, as FCDG, and draw FA, AG. And because F is the centre of the circle ABC, AF is equal to FC: also because G is the centre of the circle ADE, AG is equal to GD: therefore FA, AG are equal to FC, DG; wherefore the whole FG is greater than FA, AG: but it is also less (20. 1.); which is impossible: therefore the straight line which joins the points F, G shall not pass otherwise than through the point of contact A, that is, it must pass through it. Therefore, if two circles, &c. Q. E. D.



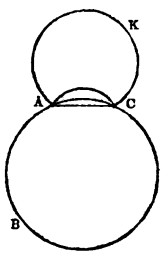
PROP. XIII. THEOR.

One circle cannot touch another in more points than one, whether it touches it on the inside or outside.

For, if it be possible, let the circle EBF touch the circle ABC in more points than one, and first on the inside, in the points B, D; join B, D, and draw (10. 11. 1.) GH bisecting BD at right angles. Therefore, because the points B, D are in the circumference of each of the circles, the straight line BD falls within each (2. 3.) of them: and their centres are (Cor. 1. 3.) in the straight line GH, which bisects BD at right angles: therefore GH passes through the point of contact (11. 3.); but it does not pass through it, because the points B, D are on the exterior of the straight line GH, which is absurd: therefore one circle cannot touch another on the inside in more points than one.



Nor can two circles touch one another on the outside in more than one point: for, if it be possible, let the circle ACK touch the circle ABC in the points A, C, and join A, C: therefore, because the two points A, C are in the circumference of the circle ACK, the straight line AC which joins them shall fall within (2. 3.) the circle ACK: and the circle ACK is on the exterior of the circle ABC; and therefore the straight line AC is on the exterior of this last circle; but because the points A, C are in the circumference of the circle ABC, the straight line AC must be within (2. 3.) the same circle, which is absurd; therefore one circle cannot touch another on the outside in more than one point: and it has been shown that they cannot touch on the inside in more points than one. Therefore, one circle, &c. Q. E. D.

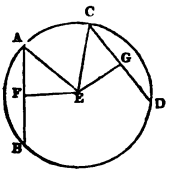


PROP. XIV. THEOR.

EQUAL straight lines in a circle, are equally distant from the centre; and those which are equally distant from the centre, are equal to one another.

Let the straight lines AB, CD, in the circle ABDC, be equal to one another: they are equally distant from the centre.

Take E the centre of the circle ABDC, and from it draw EF, EG perpendiculars to AB, CD; also join A, E and C, E. Then, because the straight line EF, passing through the centre,

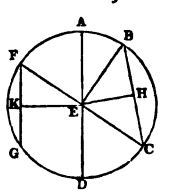


cuts the straight line AB, which does not pass through the centre, at right angles, it also bisects (3. 3.) it: wherefore AF is equal to FB, and AB double of AF. For the same reason CD is double of CG: and AB is equal to CD; therefore AF is equal to CG. And because AE is equal to EC, the square of AE is equal to the square of EC: but the squares of AF, FE are equal (47. 1.) to the square of AE, because the angle AFE is a right angle; and for the like reason, the squares of EG, GC are equal to the square of EC: therefore the squares of AF, FE are equal to the squares of CG, GE, of which the square of AF is equal to the square of CG, because AF is equal to CG; therefore the remaining square, of FE is equal to the remaining square, of EG, and the straight line EF is therefore equal to EG: but straight lines in a circle are said to be equally distant from the centre, when the perpendiculars drawn to them from the centre are equal (4. Def. 3.): therefore AB, CD are equally distant from the centre.

Next, if the straight lines AB, CD be equally distant from the centre, that is if FE be equal to EG; AB is equal to CD: for, the same construction being made, it may, as before, be demonstrated that AB is double of AF, and CD double of CG, and that the squares of EF, FA are equal to the squares of EG, GC; of which the square of FE is equal to the square of EG, because FE is equal to EG; therefore the remaining square, of AF is equal to the remaining square, of CG; and the straight line AF is therefore equal to CG: and AB is double of AF, and CD double of CG; wherefore AB is equal to CD. Therefore equal straight lines, &c. Q. E. D.

PROP. XV. THEOR.

THE diameter is the greatest straight line in a circle; and, of all others, that which is nearer to the centre is always greater than one more remote; and the greater is nearer to the centre than the less.



Let ABCD be a circle, of which the diameter is AD, and the centre E; and let BC be nearer to the centre than FG: AD is greater than any straight line BC which is not a diameter, and BC greater than FG.

From the centre draw EH, EK perpendiculars to BC, FG; also draw EB, EC, EF. And because AE is equal to EB, and ED to

EC, AD is equal to EB, EC; but EB, EC are greater (20. 1.) than BC; wherefore, also AD is greater than BC.

And because BC is nearer to the centre than FG, EH is less (5. Def. 3.) than EK: but, as was demonstrated in the preceding proposition, BC is double of BH, and FG double of FK, and the squares of EH, HB are equal to the squares of EK, KF, of which the square of EH is less than the square of EK, because EH is less than EK; therefore the square of BH is greater than the square of FK, and the straight line BH greater than FK, and therefore BC is greater than FG.

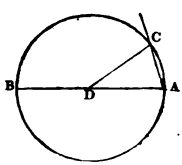
Next, let BC be greater than FG; BC is nearer to the centre than FG, that is, the same construction being made, EH is less than EK: because BC is greater than FG, BH likewise is greater than KF: and the squares of BH, HE are equal to the squares of FK, KE, of which the square of BH is greater than the square of FK, because BH is greater than FK; therefore the square of EH is less than the square of EK, and the straight line EH less than EK. Wherefore the diameter, &c. Q. E. D.

PROP. XVI. THEOR.

THE straight line drawn at right angles to the diameter of a circle, from the extremity of it, falls on the exterior of the circle; and no straight line can be drawn between that straight line and the circumference from the extremity, so as not to cut the circle; or, which is the same thing, no straight line can make so great an acute angle with the diameter at its extremity, or so small an angle with the straight line, which is at right angles to it, as not to cut the circle.

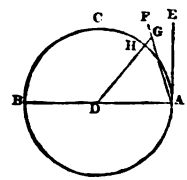
Let ABC be a circle; the centre of which is D, and the diameter AB: the straight line drawn at right angles to AB from its extremity A, shall fall on the exterior of the circle.

For, if it does not, let it fall, if possible, within the circle, as AC, and draw DC to the point C, where it meets the circumference. And because DA is equal to DC, the angle DAC is equal (5. 1.) to the angle ACD; but DAC is a right angle, therefore ACD is a right angle, and the angles DAC, ACD are therefore equal to two right angles, which is impos-



sible (17. 1.): therefore the straight line drawn from A at right angles to BA does not fall within the circle. In the same manner it may be demonstrated that it does not fall upon the circumference; therefore it must fall on the exterior of the circle, as AE.

And between the straight line AE and the circumference no straight line can be drawn from the point A which does not cut the circle: for, if possible, let FA be between them, and from the point D draw (12. 1.) DG perpendicular to FA, and let it meet the circumference in H. And because AGD is a right angle, and DAG less (17. 1.) than a right angle, DA is greater (19. 1.) than DG: but DA is equal to DH: therefore DH is greater than DG, the less than the greater, which is impossible: therefore no straight line can be drawn from the point A between AE and the circumference, which does not cut the circle; or, which amounts to the same thing, however great an acute angle a straight line makes with the diameter at the point A, or however small an angle it makes with AE, the circumference passes between that straight line and the perpendicular AE. Q. E. D.



Cor. From this it is manifest, that the straight line which is drawn at right angles to the diameter of a circle from the extremity of it, touches the circle; and that it touches it only in one point, because, if it did meet the circle in two, it would fall within it (2. 3.). 'Also it is evident that there 'can be but one straight line which touches the circle in the 'same point.'

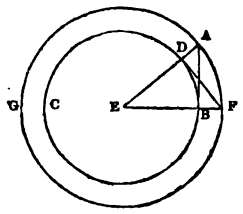
PROP. XVII. PROB.

To draw a straight line from a given point, either on the exterior of, or in the circumference, which shall touch a given circle.

First, let A be a given point on the exterior of the given circle BCD: it is required to draw a straight line from A which shall touch the circle.

Find (1. 3.) the centre E of the circle, and draw EA cutting the circle BCD in D; also from the centre E, at the distance EA, describe the circle AFG; from the point D draw (11. 1.) DF at right angles to EA, draw EF, cutting the circle BCD in B, and join A, B: AB touches the circle BCD.

Because E is the centre of the circles, BCD, AFG, EA is



equal to EF, and ED to EB; therefore the two sides, AE, EB are equal to the two FE, ED, and they contain the angle at E common to the two triangles AEB, FED; therefore the base DF is equal to the base AB; and the triangle FED to the triangle AEB, and the other angles to the other angles (4. 1.): therefore the angle EBA is equal to the angle EDF: but EDF is a right angle, wherefore EBA is a right angle; and EB is drawn from the centre: but a straight line drawn from the extremity of a diameter, at right angles to it, touches the circle (Cor. 16. 3.); therefore AB touches the circle; and it is drawn from the given point A. Q. E. F.

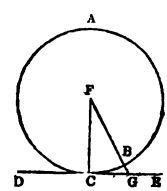
But if the given point be in the circumference of the circle, as the point D, draw DE to the centre E, and DF at right angles to DE: DF touches the circle (Cor. 16. 3.).

PROP. XVIII. THEOR.

If a straight line touches a circle, the straight line drawn from the centre to the point of contact, shall be perpendicular to the line touching the circle.

Let the straight line DE touch the circle ABC in the point C; take the centre F, and draw the straight line FC: FC is perpendicular to DE.

For, if it be not, from the point F draw FBG perpendicular to DE. And because FGC is a right angle, GCF is (17. 1.) an acute angle; and to the greater angle the greatest (19. 1.) side is opposite: therefore FC is greater than FG: but FC is equal to FB; therefore FB is greater than FG, the less than the greater, which is impossible: wherefore FG is not perpendicular to DE. In the same manner it may be shown that no other line is perpendicular to it besides FC, that is, FC is perpendicular to DE. Therefore, if a straight line, &c. Q. E. D.

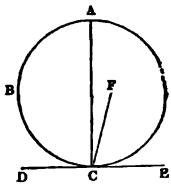


PROP. XIX. THEOR.

If a straight line touches a circle, and from the point of contact a straight line be drawn at right angles to the touching line, the centre of the circle shall be in that line.

Let the straight line DE touch the circle ABC in C, and from C let CA be drawn at right angles to DE: the centre of the circle is in CA.

For, if not, let F be the centre, if possible, and join C, F: Because DE touches the circle ABC, and FC is drawn from the centre to the point of contact, FC is perpendicular (18. 3.) to DE; therefore FCE is a right angle: but ACE is also a right angle: therefore the angle FCE is equal to the angle ACE, the less to the greater, which is impossible: wherefore F is not the centre of the circle ABC. In the same manner, it may be shown that no other point which is not in CA, is the centre; that is, the centre is in CA. Therefore, if a straight line, &c. Q. E. D.

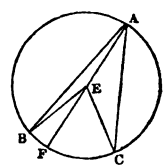


PROP. XX. THEOR.

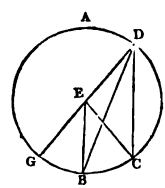
THE angle at the centre of a circle is double of the angle at the circumference, upon the same base, that is, upon the same part of the circumference.

Let ABC be a circle, and BEC an angle at the centre, and BAC an angle at the circumference, which have the same circumference BC for their base: the angle BEC is double of the angle BAC.

First, let E, the centre of the circle, be within the angle BAC; draw AE, and produce it to F. Because EA is equal to EB, the angle EAB is equal (5. 1.) to the angle EBA; therefore the angles EAB, EBA are double of the angle EAB; but the angle BEF is equal (32. 1.) to the angles EAB, EBA; therefore also the angle BEF is double of the angle EAB: for the same reason, the angle FEC is double of the angle EAC: therefore (2. Ax.) the whole angle BEC is double of the whole angle BAC.

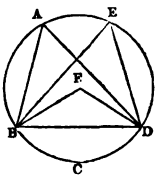


Again, let E, the centre of the circle, be on the exterior of the angle BDC: draw DE and produce it to G. It may be demonstrated, as in the first case, that the angle GEC is double of the angle GDC, and that GEB, a part of the first, is double of GDB, a part of the other; therefore (3. Ax.) the remaining angle BEC is double of the remaining angle BDC. Therefore the angle at the centre, &c. Q. E. D.



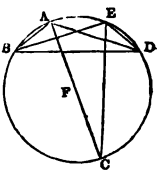
PROP. XXI. THEOR.

THE angles in the same segment of a circle are equal to one another.



Let ABCD be a circle, and BAD, BED angles in the same segment BAED: the angles BAD, BED are equal to one another.

Take F, the centre of the circle ABCD. and, first, let the segment BAED be greater than a semicircle; draw BF and FD. And because the angle BFD is at the centre, and the angle BAD at the circumference, and that they have the same part of the circumference, viz. BCD for their base; therefore the angle BFD is double (20. 3.) of the angle BAD: for the same reason, the angle BFD is double of the angle BED: therefore the angle BAD is equal to the angle BED.



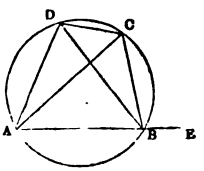
But, if the segment BAED be not greater than a semicircle, let BAD, BED be angles in it; these also are equal to one another: draw AF to the centre, and produce it to C, and join C, E: therefore the segment BADC is greater than a semicircle; and the angles in it BAC, BEC are equal, by the first case: for the same reason, because CBED is greater than a semicircle, the angles CAD, CED are equal: therefore the whole angle BAD is equal to the whole angle BED. Wherefore the angles in the same segment, &c. Q. E. D.

PROP. XXII. THEOR.

THE opposite angles of any quadrilateral figure inscribed in a circle, are together equal to two right angles.

Let ABCD be a quadrilateral figure in the circle ABCD: any two of its opposite angles are together equal to two right angles.

Join A and C, B and D; and because the three angles of every triangle are equal (32. 1.) to two right angles, the three angles of the triangle CAB, viz. the angles CAB, ABC, BCA are equal to two right angles: but the angle CAB is equal (21. 3.) to the angle CDB, because they are in the same segment BADC: and the angle ACB is equal to the angle ADB, because they are in the same seg-



ment ADCB: therefore the whole angle ADC is equal to the angles CAB, ACB. To each of these equals add the angle ABC; therefore the angles ABC, CAB, BCA are equal to the angles ABC, ADC: but ABC, CAB, BCA are equal to two right angles: therefore also the angles ABC, ADC are equal to two right angles. In the same manner, the angles BAD, DCB may be shown to be equal to two right angles. Therefore, the opposite angles, &c. Q. E. D.

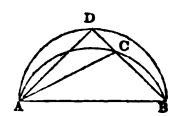
COR. If any one of the sides be produced, the exterior angle between the part produced and the adjacent side of the quadrilateral is equal to the interior and opposite angle.

Let AB be produced to E; the angle CBE will be equal to the angle ADC: for the angle CBE is equal to the sum of the angles CAB, ACB (32. 1.), and these angles together are equal to ADC by the proposition. Therefore, &c.

PROP. XXIII. THEOR.

UPON the same straight line, and upon the same side of it, there cannot be two similar segments of circles, not coinciding with one another.

If it be possible, let the two similar segments of circles, viz. ACB, ADB be upon the same side of the same straight line AB, not coinciding with one another. Then, because the circle ACB cuts the circle ADB in the two points A, B, they cannot cut one another in any other point (10. 3.): one of the segments must therefore fall within the other. Let ACB fall within ADB, and draw the straight line BCD, also join C and A, D and A: and because the segment ACB is similar to the segment ADB, and that similar segments of circles contain (11. Def. 3.) equal angles; the angle ACB is equal to the angle ADB, the exterior to the interior, which is impossible (16. 1.) Therefore, there cannot be two similar segments of a circle upon the same side of the same line, which do not coincide. Q. E. D.



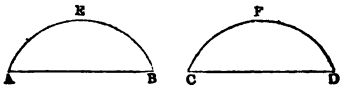
PROP. XXIV. THEOR.

SIMILAR segments of circles upon equal straight lines, are equal to one another.

Let AEB, CFD be similar segments of circles upon the equal straight lines AB, CD: the segment AEB is equal to the segment CFD.

For if the segment AEB be applied to the segment CFD, so

that the point A be on C, and the straight line AB upon CD, the point B shall coincide with the point D, because AB is equal to CD; therefore the straight line AB coinciding with CD, the segment AEB must (23. 3.) coincide with the segment CFD, and therefore is equal to it. Wherefore similar segments, &c. Q. E. D.

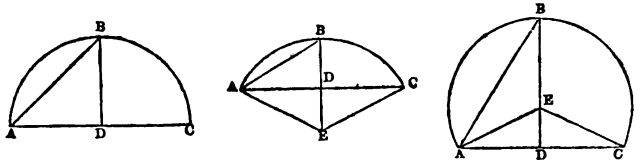


PROP. XXV. PROB.

A SEGMENT of a circle being given, to describe the circle of which it is the segment.

Let ABC be the given segment of a circle: it is required to describe the circle of which it is the segment.

Bisect (10. 1.) AC in D, and from the point D draw (11. 1.) DB at right angles to AC, and join A, B: First, let the angles ABD, BAD be equal to one another; then the straight line BD is equal (6. 1.) to DA, and therefore to DC; and because the three straight lines DA, DB, DC are all equal; D is the centre of the circle (9. 3.). From the centre D, at the distance of any of the three DA, DB, DC, describe a circle; this shall pass through the other points; and the circle of which ABC is a segment is described. And because the centre D is in AC, the segment ABC is a semicircle: but if the angles ABD, BAD are not equal to one another, at the point A, in the straight line AB, make (23. 1.) the angle BAE equal to the angle ABD, and produce BD, if necessary, to E, and join E, C. And because the angle ABE is equal to the angle BAE, the



straight line BE is equal (6. 1.) to EA: And because AD is equal to DC, and DE common to the triangles ADE, CDE, the two sides AD, DE are equal to the two CD, DE, each to each; and the angle ADE is equal to the angle CDE, for each of them is a right angle; therefore the base AE is equal (4. 1.) to the base EC: but AE was shown to be equal to EB, wherefore also BE is equal to EC: and the three straight lines AE, EB, EC are therefore equal to one another; wherefore (9. 3.) E is the centre of the circle. From the centre E, at the distance of any of the three AE, EB, EC, describe a

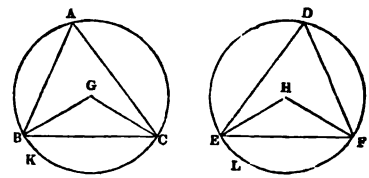
circle, this shall pass through the other points; and the circle of which ABC is a segment is described: and it is evident, that if the angle ABD be greater than the angle BAD, the centre E falls on the exterior of the segment ABC, which therefore is less than a semicircle: but if the angle ABD be less than BAD, the centre E falls within the segment ABC, which is therefore greater than a semicircle: wherefore a segment of a circle being given, the circle is described of which it is a segment. Q. E. F.

PROP. XXVI. THEOR.

In equal circles, equal angles stand upon equal circumferences, whether they be at the centres or circumferences.

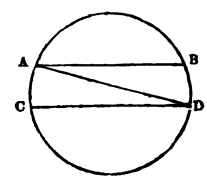
Let ABC, DEF be equal circles, and the equal angles BGC, EHF be at their centres; also let the equal angles BAC, EDF be at their circumferences: the circumference BKC is equal to the circumference ELF.

Join B and C, E and F; and because the circles ABC, DEF are equal, the straight lines drawn from their centres are equal: therefore the two sides BG, GC are equal to the two EH, HF; and the angle at G is equal to the angle at H; therefore the base BC is equal (4. 1.) to the base EF. And because the angle at A is equal to the angle at D, the segment BAC is similar (11. Def. 3.) to the segment EDF; and they are upon equal straight lines BC, EF; but similar segments of circles upon equal straight lines are equal (24. 3.) to one another, therefore the segment BAC is equal to the segment EDF: but the whole circle ABC is equal to the whole DEF; therefore the remaining segment BKC is equal to the remaining segment ELF, and the circumference BKC to the circumference ELF. Wherefore, in equal circles, &c. Q. E. D.



Cor. If, in a circle, lines be drawn parallel to one another the arcs, or parts of the circumference, which they cut off shall also be equal to one another.

Let AB and CD be two parallel lines in the circle ACDB; the arc AC shall be equal to the arc BD. For, join A, D: since AB is parallel to CD, the alternate angles BAD, ADC (at the circumference of the circle) are equal to one another



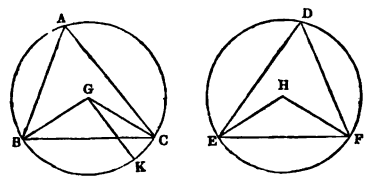
(29. 1.). Therefore, by this proposition, the circumference BD is equal to the circumference AC.

PROP. XXVII. THEOR.

In equal circles, the angles which stand upon equal circumferences are equal to one another, whether they be at the centres or circumferences.

Let the angles BGC, EHF at the centres, and BAC, EDF at the circumferences of the equal circles ABC, DEF stand upon the equal circumferences BC, EF: the angle BGC is equal to the angle EHF, and the angle BAC to the angle EDF.

If the angle BGC be not equal to the angle EHF, one of them is the greater: Let BGC be the greater, and at the point G, in the straight line BG, make (23. 1.) the angle BGK equal to the angle EHF: but equal angles stand upon equal circumferences (26. 3.), when they are at the centre; therefore the circumference BK is equal to the circumference EF: but EF is equal to BC; therefore also BK is equal to BC, the less to the greater, which is impossible: therefore the angle BGC is not unequal to the angle EHF; that is, it is equal to it: and the angle at A (20. 3.) is half of the angle BGC, and the angle at D half of the angle EHF: therefore the angle at A is equal to the angle at D. Wherefore, in equal circles, &c. Q. E. D.

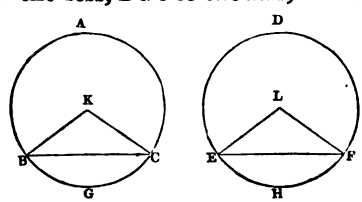


PROP. XXVIII. THEOR.

In equal circles, equal straight lines cut off equal circumferences, the greater equal to the greater, and the less to the less.

Let ABC, DEF be equal circles, and BC, EF equal straight lines in them, which cut off the two greater circumferences, BAC, EDF, and the two less, BGC, EHF: the greater, BAC is equal to the greater, EDF, and the less, BGC to the less, EHF.

Take (1. 3.) K, L, the centres of the circles, and draw BK, KC, EL, LF. And because the circles are equal, the straight lines from their centres are equal; therefore BK, KC are equal to EL, LF; and the



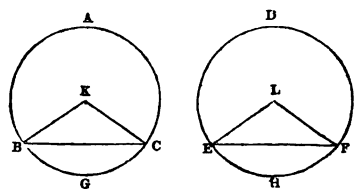
base BC is equal to the base EF: therefore the angle BKC is equal (8. 1.) to the angle ELF: but equal angles stand upon equal (26. 3.) circumferences, when they are at the centres; therefore the circumference BGC is equal to the circumference EHF. But the whole circle ABC is equal to the whole EDF; the remaining part therefore of the circumference, viz. BAC, is equal to the remaining part EDF. Therefore, in equal circles, &c. Q. E. D.

PROP. XXIX. THEOR.

IN equal circles, equal circumferences are subtended by equal straight lines.

Let ABC, DEF be equal circles, and let the circumferences BGC, EHF also be equal; and draw BC, EF: the straight line BC is equal to the straight line EF.

Take (1. 3,) K, L, the centres of the circles, and draw BK, KC, EL, LF. And because the circumference BGC is equal to the circumference EHF, the angle BKC is equal (27. 3.) to the angle ELF. And because the circles ABC, DEF, are equal, the straight lines from their centres are equal: therefore BK, KC are equal to EL, LF, and they contain equal angles: therefore the base BC is equal (4. 1.) to the base EF. Therefore, in equal circles, &c. Q. E. D.



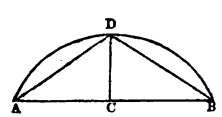
PROP. XXX. PROB.

To bisect a given circumference, *that is, to divide it into two equal parts.*

Let ADB be the given circumference: it is required to bisect it.

Join A, B, and bisect (10. 1.) it in C; from the point C draw CD at right angles to AB, and join A and D, D and B: the circumference ADB is bisected in the point D.

Because AC is equal to CB, and CD common to the triangles ACD, BCD, the two sides, AC, CD are equal to the two BC, CD; and the angle ACD is equal to the angle BCD, because each of them is a right angle: therefore the base AD is equal (4. 1.) to the base BD. But equal straight lines cut off equal (28. 2.)



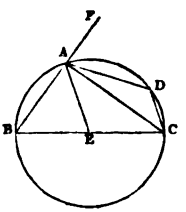
circumferences, the greater equal to the greater, and the less to the less, and AD, DB are each of them less than a semicircle; because DC passes through the centre (Cor. 1. 3.): wherefore the circumference AD is equal to the circumference DB; therefore the given circumference is bisected in D. Q.E.F.

PROP. XXXI. THEOR.

In a circle, the angle in a semicircle is a right angle; but the angle in a segment greater than a semicircle is less than a right angle; and the angle in a segment less than a semicircle is greater than a right angle.

Let ABCD be a circle, of which the diameter is BC, and centre E; and draw CA, dividing the circle into the segments ABC, ADC, and draw BA, AD, DC: the angle in the semicircle BAC is a right angle; and the angle in the segment ABC, which is greater than a semicircle, is less than a right angle; and the angle in the segment ADC, which is less than a semicircle, is greater than a right angle.

Join A, E, and produce BA to F. And because BE is equal to EA, the angle EAB is equal (5. 1.) to EBA; also, because AE is equal to EC, the angle EAC is equal to ECA; wherefore the whole angle BAC is equal to the two angles ABC, ACB: but FAC, the exterior angle of the triangle ABC, is equal (32. 1.) to the two angles ABC, ACB; therefore the angle BAC is equal to the angle FAC, and each of them is therefore a right (10. Def. 1.) angle: wherefore the angle BAC in a semicircle is a right angle.



And because the two angles ABC, BAC of the triangle ABC are together less (17. 1.) than two right angles, and that BAC is a right angle, ABC must be less than a right angle; and therefore the angle in a segment ABC greater than a semi-circle, is less than a right angle.

And because ABCD is a quadrilateral figure in a circle, any two of its opposite angles are equal (22. 3.) to two right angles; therefore the angles ABC, ADC are equal to two right angles; and ABC is less than a right angle; wherefore the other, ADC is greater than a right angle.

N.B. From this proposition is derived the following practical method of drawing from a given point in a given line, at or near one of its extremities, a perpendicular to that line.

Let AB be the given line and C the given point: take any convenient

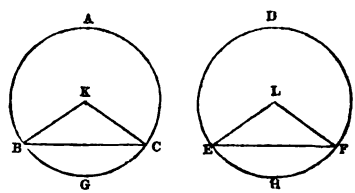
base BC is equal to the base EF: therefore the angle BKC is equal (8. 1.) to the angle ELF: but equal angles stand upon equal (26. 3.) circumferences, when they are at the centres; therefore the circumference BGC is equal to the circumference EHF. But the whole circle ABC is equal to the whole EDF; the remaining part therefore of the circumference, viz. BAC, is equal to the remaining part EDF. Therefore, in equal circles, &c. Q. E. D.

PROP. XXIX. THEOR.

In equal circles, equal circumferences are subtended by equal straight lines.

Let ABC, DEF be equal circles, and let the circumferences BGC, EHF also be equal; and draw BC, EF: the straight line BC is equal to the straight line EF.

Take (1. 3,) K, L, the centres of the circles, and draw BK, KC, EL, LF. And because the circumference BGC is equal to the circumference EHF, the angle BKC is equal (27. 3.) to the angle ELF. And because the circles ABC, DEF, are equal, the straight lines from their centres are equal: therefore BK, KC are equal to EL, LF, and they contain equal angles: therefore the base BC is equal (4. 1.) to the base EF. Therefore, in equal circles, &c. Q. E. D.



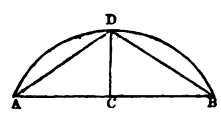
PROP. XXX. PROB.

To bisect a given circumference, *that is, to divide it into two equal parts.*

Let ADB be the given circumference: it is required to bisect it.

Join A, B, and bisect (10. 1.) it in C; from the point C draw CD at right angles to AB, and join A and D, D and B: the circumference ADB is bisected in the point D.

Because AC is equal to CB, and CD common to the triangles ACD, BCD, the two sides, AC, CD are equal to the two BC, CD; and the angle ACD is equal to the angle BCD, because each of them is a right angle: therefore the base AD is equal (4. 1.) to the base BD. But equal straight lines cut off equal (28. 2.)



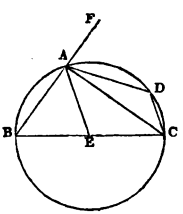
circumferences, the greater equal to the greater, and the less to the less, and AD, DB are each of them less than a semicircle; because DC passes through the centre (Cor. 1. 3.): wherefore the circumference AD is equal to the circumference DB; therefore the given circumference is bisected in D. Q.E.F.

PROP. XXXI. THEOR.

In a circle, the angle in a semicircle is a right angle; but the angle in a segment greater than a semicircle is less than a right angle; and the angle in a segment less than a semicircle is greater than a right angle.

Let ABCD be a circle, of which the diameter is BC, and centre E; and draw CA, dividing the circle into the segments ABC, ADC, and draw BA, AD, DC: the angle in the semicircle BAC is a right angle; and the angle in the segment ABC, which is greater than a semicircle, is less than a right angle; and the angle in the segment ADC, which is less than a semicircle, is greater than a right angle.

Join A, E, and produce BA to F. And because BE is equal to EA, the angle EAB is equal (5. 1.) to EBA; also, because AE is equal to EC, the angle EAC is equal to ECA; wherefore the whole angle BAC is equal to the two angles ABC, ACB: but FAC, the exterior angle of the triangle ABC, is equal (32. 1.) to the two angles ABC, ACB; therefore the angle BAC is equal to the angle FAC, and each of them is therefore a right (10. Def. 1.) angle: wherefore the angle BAC in a semicircle is a right angle.



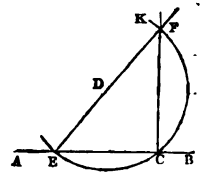
And because the two angles ABC, BAC of the triangle ABC are together less (17. 1.) than two right angles, and that BAC is a right angle, ABC must be less than a right angle; and therefore the angle in a segment ABC greater than a semi-circle, is less than a right angle.

And because ABCD is a quadrilateral figure in a circle, any two of its opposite angles are equal (22. 3.) to two right angles; therefore the angles ABC, ADC are equal to two right angles; and ABC is less than a right angle; wherefore the other, ADC is greater than a right angle.

N.B. From this proposition is derived the following practical method of drawing from a given point in a given line, at or near one of its extremities, a perpendicular to that line.

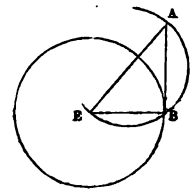
Let AB be the given line and C the given point: take any convenient

point D, and from D as a centre, with a radius equal to a line supposed to be drawn from D to C, describe an arc as ECK greater than a semi-circumference: draw a line through E and D, and produce it till it cuts the circle in F; also join C, F: CF is the perpendicular required. For since EDF passes through D, the centre of the circle, the segment ECF is a semicircle, and by the proposition, the angle ECF is a right angle.



On this proposition depends also the practical method of drawing a line to touch a circle, from a given point on its exterior.

Let A be the given point, and E the centre of the circle: join A, E; on AE describe a semi-circle cutting the given circle in B, and join A, B: AB is the line required. For join B, E; then EB is part of a diameter, and the angle ABE being in a semicircle is a right angle by the proposition: therefore AB (16. 3.) touches the circle at the point B.

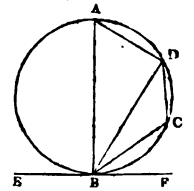


PROP. XXXII. THEOR.

If a straight line touches a circle, and from the point of contact a straight line be drawn cutting the circle; the angles made by this line with the line touching the circle, shall be equal to the angles which are in the alternate segments of the circle.

Let the straight line EF touch the circle ABCD in B, and from the point B let the straight line BD be drawn, cutting the circle: the angles which BD makes with the touching line EF shall be equal to the angles in the alternate segments of the circle: that is, the angle FBD is equal to the angle which is in the segment DAB, and the angle DBE to the angle in the segment BCD.

From the point B draw (11. 1.) BA at right angles to EF; take any point C in the circumference BD, and draw AD, DC, CB. And because the straight line EF touches the circle ABCD in the point B, and BA is drawn at right angles to the touching line from the point of contact B, the centre of the circle is (19. 3.) in BA; therefore the angle ADB in a semicircle is a right (31. 3.) angle, and consequently the two other angles BAD, ABD are equal (32. 1.) to a right angle: but ABF is likewise a right angle; therefore the angle ABF is equal to the angles BAD, ABD. Take from these equals the common angle ABD; therefore the remaining angle DBF is equal to the angle BAD, which is in the alter-

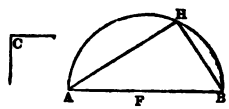


nate segment of the circle; and because ABCD is a quadrilateral figure in a circle, the opposite angles BAD, BCD are equal (22. 3.) to two right angles: therefore the angles DBF, DBE, being likewise equal (13. 1.) to two right angles, are equal to the angles BAD, BCD; and DBF has been proved equal to BAD: therefore the remaining angle DBE is equal to the angle BCD in the alternate segment of the circle. Wherefore, if a straight line, &c. Q. E. D.

PROP. XXXIII. PROB.

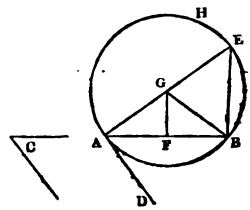
Upon a given straight line to describe a segment of a circle containing an angle equal to a given rectilineal angle.

Let AB be the given straight line, and the angle at C the given rectilineal angle: it is required to describe upon the given straight line AB a segment of a circle, containing an angle equal to the angle C.

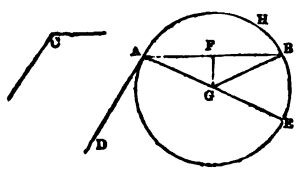


First, let the angle at C be a right angle, and bisect (10. 1.) AB in F, and from the centre F, at the distance FB, describe the semicircle AHB; then the angle AHB, being in a semicircle, is (31. 3.) equal to the right angle at C.

But, if the angle C be not a right angle, at the point A, in the straight line AB, make (23. 1.) the angle BAD equal to the angle C, and from a point A draw (11. 1.) AE at right angles to AD; bisect AB in F, and from F draw (11. 1.) FG at right angles to AB, and join G, B. And because AF is equal to FB, and FG common to the triangles AFG, BFG, the two sides AF, FG are equal to the two BF, FG; and the angle AFG is equal to the angle BFG; therefore the base AG is equal (4. 1.) to the base GB; and the circle described from the centre G, at the distance GA, shall pass through the point B: let this be the circle AHB.



And because from the point A, the extremity of the diameter AE, AD is drawn at right angles to AE, therefore AD (Cor. 16. 3.) touches the circle; and because AB, drawn from the point of contact A, cuts the circle, the angle DAB is equal to the angle in the alternate



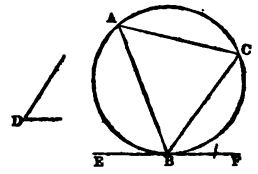
segment AHB (32. 3.): but the angle DAB is equal to the angle C, therefore also the angle C is equal to the angle in the segment AHB: wherefore, upon the given straight line AB the segment AHB of a circle is described, which contains an angle equal to the given angle at C. Q. E. F.

PROP. XXXIV. PROB.

To cut off a segment from a given circle which shall contain an angle equal to a given rectilineal angle.

Let ABC be the given circle, and D the given rectilineal angle: it is required to cut off a segment from the circle ABC that shall contain an angle equal to the given angle D.

Draw (17. 3.) the straight line EF touching the circle ABC in the point B, and at the point B, in the straight line BF, make (23. 1.) the angle FBC equal to the angle D. Therefore, because the straight line EF touches the circle ABC, and BC is drawn from the point of contact B, the angle FBC is equal (32. 3.) to the angle in the alternate segment BAC of the circle; but the angle FBC is equal to the angle D; therefore the angle in the segment BAC is equal to the angle D: wherefore the segment BAC is cut off from the given circle ABC, containing an angle equal to the given angle D. Q. E. F.

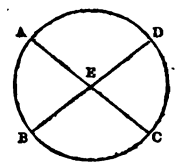


PROP. XXXV. THEOR.

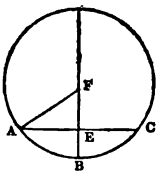
If two straight lines within a circle cut one another, the rectangle contained by the segments of one of them, is equal or equivalent to the rectangle contained by the segments of the other.

Let the two straight lines AC, BD, within the circle ABCD, cut one another in the point E: the rectangle contained by AE, EC is equal or equivalent to the rectangle contained by BE, ED.

If AC, BD pass each of them through the centre, so that E is the centre; it is evident, that AE, EC, BE, ED being all equal, the rectangle, AE, EC is likewise equal to the rectangle BE, ED.

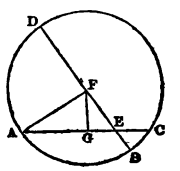


But let one of them, BD pass through the centre, and cut the other AC, which does not pass through the centre, at

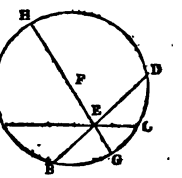


right angles, in the point E: then, if BD be bisected in F, F is the centre of the circle ABCD; join A, F. And because BD, which passes through the centre, cuts the straight line AC, which does not pass through the centre, at right angles in E, AE, EC are equal (3. 3.) to one another: and because the straight line BD is cut into two equal parts in the point F, and into two unequal parts in the point E, the rectangle BE, ED, together with the square of EF, is equivalent (5. 2.) to the square of FB; that is, to the square of FA: but the squares of AE, EF are equivalent (47. 1.) to the square of FA; therefore the rectangle BE, ED, together with the square of EF, is equivalent to the squares of AE, EF. Take away the common square of EF, and the remaining rectangle BE, ED is equivalent to the remaining square of AE; that is to the rectangle AE, EC.

Next, let BD, which passes through the centre, cut the other, AC, which does not pass through the centre, in E, but not at right angles: then, as before, if BD be bisected in F, F is the centre of the circle. Join A, F, and from F draw (12. 1.) FG perpendicular to AC; therefore AG is equal (3. 3.) to GC; wherefore the rectangle AE, EC, together with the square of EG is equivalent (5. 2.) to the square of AG. To each of these equivalents add the square of GF; therefore the rectangle AE, EC, together with the squares of EG, GF, is equivalent to the squares of AG, GF: but the squares of EG, GF are equivalent (47. 1.) to the square of EF; and the squares of AG, GF are equivalent to the square of AF: therefore the rectangle AE, EC, together with the square of EF, is equivalent to the square of AF; that is, to the square of FB: but the square of FB is equal (5. 2.) to the rectangle BE, ED, together with the square of EF; therefore the rectangle AE, EC, together with the square of EF, is equivalent to the rectangle BE, ED, together with the square of EF. Take away the common square of EF, and the remaining rectangle AE, EC is therefore equivalent to the remaining rectangle BE, ED.



Lastly, let neither of the straight lines AC, BD pass through the centre: take the centre F, and through E, the intersection of the straight lines AC, DB, draw the diameter GEFH. And because the rectangle AE, EC is equivalent, as has been shown,



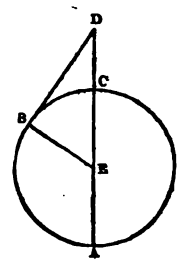
to the rectangle, GE, EH; and, for the same reason, the rectangle BE, ED is equivalent to the same rectangle GE, EH; therefore the rectangle AE, EC is equivalent to the rectangle BE, ED. Wherefore, if two straight lines, &c. Q. E. D.

PROP. XXXVI. THEOR.

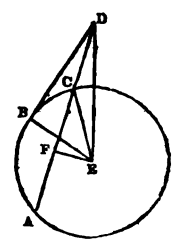
If from any point on the exterior of a circle two straight lines be drawn, one of which cuts the circle, and the other touches it; the rectangle contained by the whole line which cuts the circle, and the part of it which is on the exterior of the circle, shall be equivalent to the square of the line which touches it.

Let D be any point on the exterior of the circle ABC, and DCA, DB two straight lines drawn from it, of which DCA cuts the circle, and DB touches the same: the rectangle AD, DC is equivalent to the square of DB.

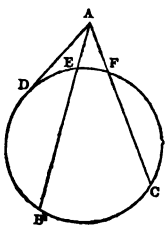
Either DCA passes through the centre, or it does not: first let it pass through the centre E, and join E, B; therefore the angle EBD is a right (18. 3.) angle. And because the straight line AC is bisected in E, and produced to the point D, the rectangle AD, DC, together with the square of EC, is equivalent (6. 2.) to the square of ED, and CE is equal to EB; therefore the rectangle AD, DC, together with the square of EB, is equivalent to the square of ED: but the square of ED is equivalent (47. 1.) to the squares of EB, BD, because EBD is a right angle; therefore the rectangle AD, DC, together with the square of EB, is equivalent to the squares of EB, BD. Take away the common square of EB; therefore the remaining rectangle AD, DC is equivalent to the square of the line DB which touches the circle.



But if DCA does not pass through the centre of the circle ABC, take (1. 3.) the centre E, and draw EF perpendicular (12. 1.) to AC; also draw EB, EC, ED. And because the straight line EF, which passes through the centre, cuts the straight line AC, which does not pass through the centre, at right angles, it shall likewise (3. 3.) bisect it: therefore AF is equal to FC: and because the straight line AC is bisected in F, and produced to D, the rectangle AD, DC, together with the square of FC, is equivalent (6. 2.) to



the square of FD. To each of these equivalents add the square of FE; therefore the rectangle AD, DC, together with the squares of CF, FE, is equivalent to the squares of DF, FE: but the square of ED is equivalent (47. 1.) to the squares of DF, FE, because EFD is a right angle; and the square of EC is equivalent to the squares of CF, FE; therefore the rectangle AD, DC, together with the square of EC, is equivalent to the square of ED: and CE is equal to EB; therefore the rectangle AD, DC, together with the square of EB, is equivalent to the square of ED: but the squares of EB, BD are equivalent to the square (47. 1.) of ED, because EBD is a right angle; therefore the rectangle AD, DC, together with the square of EB, is equivalent to the squares of EB, BD. Take away the common square of EB; therefore the remaining rectangle AD, DC is equivalent to the square of DB. Wherefore, if from any point, &c. Q. E. D.



Cor. If from any point on the exterior of a circle, there be drawn two straight lines cutting it, as AB, AC, the rectangles contained by the whole lines and the parts of them on the exterior of the circle, are equivalent to one another, viz. the rectangle BA, AE, to the rectangle CA, AF: for each of them is equivalent to the square of the straight line AD which touches the circle.

Cor. A. If from any point as A, on the exterior of a circle, there be drawn two straight lines to touch the circle, these lines shall be equal to one another. For the square of each of the lines is equivalent to the rectangle BA, AE; therefore the squares of the two lines, and consequently the lines themselves, are equal to one another.

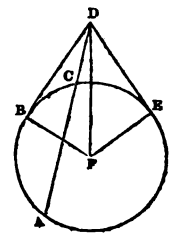
PROP. XXXVII. THEOR.

If from a point on the exterior of a circle there be drawn two straight lines, one of which cuts the circle, and the other meets it; if the rectangle contained by the whole line which cuts the circle, and the part of it which is on the exterior of the circle, be equivalent to the square of the line which meets it, the line which meets shall touch the circle.

Let any point D be taken on the exterior of the circle ABC, and from it let two straight lines DCA and DB be drawn, of which DCA cuts the circle, and DB meets it; if the

rectangle AD, DC be equivalent to the square of DB: DB touches the circle.

Draw (17. 3.) the straight line DE, touching the circle ABC, find its centre F, and draw FE, FB, FD; then FED is a right (18. 3.) angle. And because DE touches the circle ABC, and DCA cuts it, the rectangle AD, DC is equivalent (36. 3.) to the square of DE: but the rectangle AD, DC is, by hypothesis, equivalent to the square of DB: therefore the square of DE is equal to the square of DB; and the straight line DE equal to the straight line DB: and FE is equal to FB; wherefore DE, EF are equal to DB, BF; and the base FD is common to the two triangles DEF, DBF; therefore the angle DEF is equal (8. 1.) to the angle DBF; but DEF is a right angle, therefore also DBF is a right angle. And BF, if produced, is a diameter, and the straight line which is drawn at right angles to a diameter, from the extremity of it, touches (16. 3.) the circle; therefore DB touches the circle ABC. Wherefore, if from a point, &c. Q. E. D.



ELEMENTS

OF

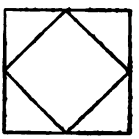
GEOMETRY.

EUCLID. BOOK IV.

DEFINITIONS.

I.

A RECTILINEAL figure is said to be inscribed in another rectilineal figure, when all the angles of the inscribed figure are upon the sides of the figure in which it is inscribed, each upon each.

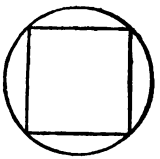


II.

In like manner, a figure is said to be described about another figure, when all the sides of the circumscribed figure pass through the angular points of the figure about which it is described, each through each.

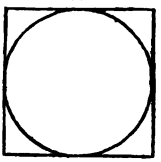
III.

A rectilineal figure is said to be inscribed in a circle, when all the angles of the inscribed figure are upon the circumference of the circle.



IV.

A rectilineal figure is said to be described about a circle, when each side of the circumscribed figure touches the circumference of the circle.



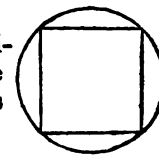
V.

In like manner, a circle is said to be inscribed in a rectilineal

figure, when the circumference of the circle touches each side of the figure.

VI.

A circle is said to be described about a rectilineal figure, when the circumference of the circle passes through all the angular points of the figure about which it is described.



VII.

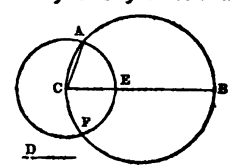
A straight line is said to be placed in a circle, when the extremities of it are in the circumference of the circle.

PROP. I. PROB.

IN a given circle to place a straight line, equal to a given straight line not greater than the diameter of the circle.

Let ABC be the given circle, and D the given straight line not greater than the diameter of the circle.

Draw BC the diameter of the circle ABC; then, if BC is equal to D, the thing required is done; for in the circle ABC a straight line BC is placed equal to D: but, if it is not, BC is greater than D; make CE equal (3. 1.) to D, and from the centre C, at the distance CE, describe the circle AEF, and join C, A: therefore, because C is the centre of the circle AEF, CA is equal to CE; but D is equal to CE; therefore D is equal to CA: wherefore in the circle ABC, a straight line is placed, equal to the given straight line D, which is not greater than the diameter of the circle. Q. E. F.



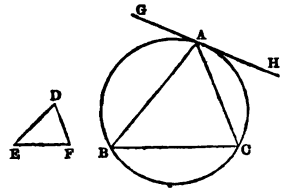
PROP. II. PROB.

IN a given circle to inscribe a triangle equiangular to a given triangle.

Let ABC be the given circle, and DEF the given triangle: it is required to inscribe in the circle ABC a triangle equiangular to the triangle DEF.

Draw (17. 3.) the straight line GAH touching the circle in

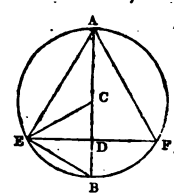
the point A, and at the point A, in the straight line AH, make (23. 1.) the angle HAC equal to the angle DEF; and at the point A, in the straight line AG, make the angle GAB equal to the angle DFE, and join B, C: therefore, because HAG touches the circle ABC, and AC is drawn from the point of contact, the angle HAC is equal (32. 3.) to the angle ABC in the alternate segment of the circle: but HAC is equal to the angle DEF; therefore also the angle ABC is equal to DEF: for the same reason, the angle ACB is equal to the angle DFE; therefore the remaining angle BAC is equal (32. 1.) to the remaining angle EDF: wherefore the triangle ABC is equiangular to the triangle DEF, and it is inscribed in the circle ABC. Q. E. F.



N.B. The following method of inscribing an equilateral triangle in a circle is frequently useful.

Let AB be a diameter of the circle, and let C be the centre. Bisect CB in D, and draw EDF at right angles to CB: join E, C and E, B; also join E, A and A, F: the triangle AEF is equilateral.

In the triangles EDC and EDB, CD is equal to BD, ED is common, and the angle EDC is equal to EDB; therefore EB is equal to EC or CB, and the triangle ECB is equilateral. In like manner it may be shown, supposing C, F and B, F to be joined, that the triangle BCF is equilateral; therefore the angle BCE is equal to the angle BCF, and each of them is equal to one third of two right angles (32. 1.), also the angle ECF is equal to two thirds of two right angles. Taking each of the angles BCE, BCF, from two right angles, each of the remaining angles ACE, ACF will be equal to two thirds of two right angles; therefore the three angles ECF, ECA, ACF will be equal to one another. But the sides which contain them, viz. CE, CF, CA are equal to one another, being radii of the circle; therefore (4. 1.) EF, EA, AF will be equal to one another, and the triangle AEF is equilateral.



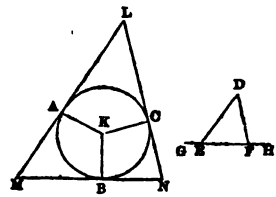
PROP. III. PROB.

ABOUT a given circle to describe a triangle equiangular to a given triangle.

Let ABC be the given circle, and DEF the given triangle: it is required to describe a triangle about the circle ABC equiangular to the triangle DEF.

Produce EF both ways to the points, G, H, and find the centre K of the circle ABC, and from it draw any straight

line KB: at the point K, in the straight line KB, make (23. 1.) the angle BKA equal to the angle DEG, and the angle BKC equal to the angle DFH; and through the points A, B, C draw the straight lines LAM, MBN, NCL, touching (17. 3.) the circle ABC. Therefore, because LM, MN, NL touch the circle ABC in the points A, B, C, to which from the centre are drawn KA, KB, KC, the angles at the points A, B, C, are right (18. 3.) angles. And because the four angles of the quadrilateral figure AMBK are equal to four right angles, for it can be divided into two triangles; and that two of them, KAM, KBM are right angles, the other two AKB, AMB are equal to two right angles: but the angles DEG, DEF are likewise equal (13. 1.) to two right angles; therefore the angles AKB, AMB are equal to the angles DEG, DEF, of which AKB is equal to DEG; wherefore the remaining angle AMB is equal to the remaining angle DEF. In like manner the angle LNM may be demonstrated to be equal to DFE; and therefore the remaining angle MLN is equal (32. 1.) to the remaining angle EDF: wherefore the triangle LMN is equiangular to the triangle DEF: and it is described about the circle ABC. Q. E. F.

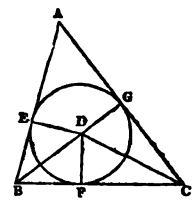


PROP. IV. PROB.

To inscribe a circle in a given triangle.

Let the given triangle be ABC: it is required to inscribe in it a circle.

Bisect (9. 1.) the angles ABC, BCA by the straight lines BD, CD meeting one another in the point D, from which draw (12. 1.) DE, DF, DG perpendiculars to AB, BC, CA. And because the angle EBD is equal to the angle FBD, for the angle ABC is bisected by BD, and that the right angle BED is equal to the right angle BFD, the two triangles EBD, FBD have two angles of the one equal to two angles of the other, and the side BD, which is opposite to one of the equal angles in each, is common to both; therefore their other sides shall be equal (26. 1.); wherefore DE is equal to DF: for the same reason, DG is equal to DF; therefore the three straight lines DE, DF, DG



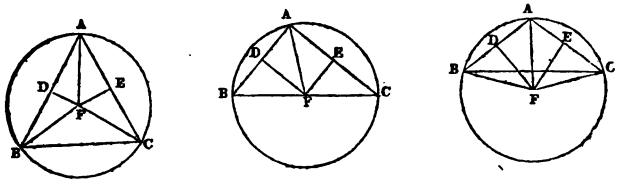
are equal to one another, and the circle described from the centre D, at the distance of any of them, shall pass through the extremities of the two others, and it shall touch the straight lines AB, BC, CA, because the angles at the points E, F, G are right angles, and the straight line which is drawn from the extremity of a diameter at right angles to it, touches (16. 3.) the circle; therefore the straight lines AB, BC, CA do each of them touch the circle, and the circle EFG is inscribed in the triangle ABC. Q. E. F.

PROP. V. PROB.

To describe a circle about a given triangle.

Let the given triangle be ABC: it is required to describe about it a circle.

Bisect (10. 1.) AB, AC, in the points D, E, and from these points draw DF, EF at right angles (11. 1.) to AB, AC; DF,



EF produced meet one another. For, if they do not meet, they are parallel, wherefore AB, AC, which are at right angles to them, are parallel; which is absurd; let them meet in F, and join F, A: also if the point F be not in BC, join B, F and C, F: then, because AD is equal to DB, and DF common, and at right angles to AB, the base AF is equal (4. 1.) to the base FB. In like manner it may be shown that CF is equal to FA; and therefore BF is equal to FC; and FA, FB, FC are equal to one another: wherefore the circle described from the centre F, at the distance of one of them, shall pass through the extremities of the two others, and be described about the triangle ABC. Q. E. F.

Cor. And it is manifest, that when the centre of the circle falls within the triangle, each of its angles is less than a right angle, each of them being in a segment greater than a semicircle; but, when the centre is in one of the sides of the triangle, the angle opposite to this side, being in a semicircle, is a right angle; and, if the centre falls out of the triangle, the angle opposite to the side beyond which it is, being in a segment less than a semicircle, is greater than a

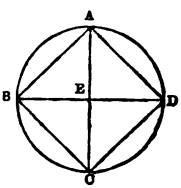
right angle: wherefore, if the given triangle be acute-angled, the centre of the circle falls within it; if it be a right-angled triangle, the centre is in the side opposite to the right angle; and, if it be an obtuse-angled triangle, the centre falls out of the triangle, beyond the side opposite to the obtuse angle.

PROP. VI. PROB.

To inscribe a square in a given circle.

Let ABCD be the given circle: it is required to inscribe in it a square.

Draw the diameters AC, BD, at right angles to one another; also draw AB, BC, CD, DA. Because BE is equal to ED, for E is the centre, and that EA is common, and at right angles to BD; the base BA is equal (4. 1.) to the base AD: and, for the same reason, BC, CD are each of them equal to BA, or AD; therefore the quadrilateral figure ABCD is equilateral. It is also rectangular; for the straight line BD, being the diameter of the circle ABCD, BAD is a semicircle; wherefore the angle BAD is a right (31. 3.) angle: for the same reason, each of the angles ABC, BCD, CDA is a right angle; therefore the quadrilateral figure ABCD is rectangular, and it has been shown to be equilateral; therefore it is a square; and it is inscribed in the circle ABCD. Q. E. F.

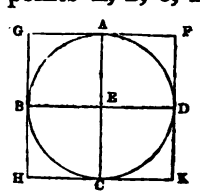


PROP. VII. PROB.

To describe a square about a given circle.

Let ABCD be the given circle: it is required to describe about it a square.

Draw two diameters AC, BD of the circle ABCD, at right angles to one another, and through the points A, B, C, D draw (17. 3.) FG, GH, HK, KF touching the circle. And because FG touches the circle ABCD, and EA is drawn from the centre E to the point of contact A, the angles at A are right (18. 3.) angles: for the same reason, the angles at the points B, C, D are right angles: and because the angle AEB is a right angle, as likewise is EBG, GH is paral-



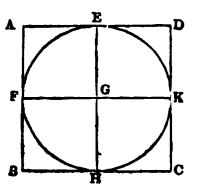
lel (28. 1.) to AC: for the same reason, AC is parallel to FK, and in like manner GF, HK may each of them be demonstrated to be parallel to BED; therefore the figures GK, GC, AK, FB, BK are parallelograms; and GF is therefore equal (34. 1.) to HK, and GH to FK: and because AC is equal to BD, and that AC is equal to each of the two GH, FK; and BD to each of the two GF, HK; GH, FK are each of them equal to GF or HK; therefore the quadrilateral figure FGHK is equilateral. It is also rectangular; for GBEA being a parallelogram, and AEB a right angle, AGB (34. 1.) is likewise a right angle. In the same manner it may be shown that the angles at H, K, F are right angles; therefore the quadrilateral figure FGHK is rectangular, and it was demonstrated to be equilateral; therefore it is a square; and it is described about the circle ABCD. Q. E. F.

PROP. VIII. PROB.

To inscribe a circle in a given square.

Let ABCD be the given square: it is required to inscribe in it a circle.

Bisect (10. 1.) each of the sides AB, AD, in the points F, E, and through E draw (31. 1.) EH parallel to AB or DC, and through F draw FK parallel to AD or BC; therefore each of the figures AK, KB, AH, HD, AG, GC, BG, GD is a parallelogram, and their opposite sides are equal (34. 1.): and because AD is equal to AB, and that AE is the half of AD, and AF the half of AB, AE is equal to AF; wherefore the sides opposite to these are equal, viz. FG to GE. In the same manner it may be demonstrated that GH, GK are each of them equal to FG or GE: therefore the four straight lines GE, GF, GH, GK are equal to one another; and the circle described from the centre G, at the distance of one of them, shall pass through the extremities of the three others. It shall also touch the straight lines AB, BC, CD, DA; because the angles at the points E, F, H, K are right (29. 1.) angles, and that the straight line which is drawn from the extremity of a diameter, at right angles to it, touches the circle (16. 3.); therefore each of the straight lines AB, BC, CD, DA touches the circle, which therefore is inscribed in the square ABCD. Q. E. F.

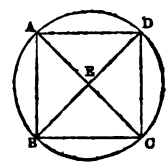


PROP. IX. PROB.

To describe a circle about a given square.

Let ABCD be the given square: it is required to describe a circle about it.

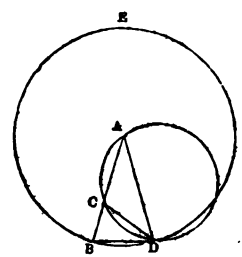
Draw AC, BD, cutting one another in E; and because DA is equal to AB, and AC common to the triangles DAC, BAC, the two sides, DA, AC are equal to the two BA, AC, and the base DC is equal to the base BC; wherefore the angle DAC is equal (8. 1.) to the angle BAC, and the angle DAB is bisected by the straight line AC. In the same manner it may be demonstrated that the angles, ABC, BCD, CDA are severally bisected by the straight lines BD, AC; therefore, because the angle DAB is equal to the angle ABC, and that the angle EAB is the half of DAB, and EBA the half of ABC; the angle EAB is equal to the angle EBA; wherefore the side EA is equal (6. 1.) to the side of EB. In the same manner it may be demonstrated that the straight lines EC, ED are each of them equal to EA or EB; therefore the four straight lines EA, EB, EC, ED are equal to one another: and the circle described from the centre E, at the distance of one of them, shall pass through the extremities of the three others, and be described about the square ABCD. Q. E. F.



PROP. X. PROB.

To describe an isosceles triangle, having each of the angles at the base double of the third angle.

Take any straight line AB, and divide it (11. 2.) in the point C, so that the rectangle AB, BC be equivalent to the square of CA; and from the centre A, at the distance AB, describe the circle BDE, in which, place (1. 4.) the straight line BD equal to AC, which is not greater than the diameter of the circle BDE; draw DA, DC, and about the triangle ADC describe (5. 4.) the circle ACD: the triangle ABD is such as is required, that is, each of the angles ABD, ADB is double of the angle BAD.



Because the rectangle AB, BC is equivalent to the square of AC, and that AC is equal to BD, the rectangle AB, BC is equivalent to the square of BD. And because from the point B, on the exterior of the circle ACD, two straight lines BCA, BD are drawn to the circumference, one of which cuts, and the other meets the circle, and that the rectangle AB, BC, contained by the whole of the cutting line, and the part of it on the exterior of the circle is equivalent to the square of BD, which meets it; the straight line BD touches (37. 3.) the circle ACD: and because BD touches the circle, and DC is drawn from the point of contact D, the angle BDC is equal (32. 3.) to the angle DAC in the alternate segment of the circle; to each of these add the angle CDA; therefore the whole angle BDA is equal to the two angles CDA, DAC; but the exterior angle BCD is equal (32. 1.) to the angles CDA, DAC; therefore also BDA is equal to BCD: but BDA is equal (5. 1.) to the angle CBD, because the side AD is equal to the side AB; therefore CBD, or DBA, is equal to BCD; and consequently the three angles BDA, DBA, BCD are equal to one another: and because the angle DBC is equal to the angle BCD, the side BD is equal (6. 1.) to the side DC; but BD was made equal to CA, therefore also CA is equal to CD, and the angle CDA equal (5. 1.) to the angle DAC; therefore the angles CDA, DAC together are double of the angle DAC: but BCD is equal to the angles CDA, DAC; therefore also BCD is double of DAC, and BCD is equal to each of the angles BDA, DBA; each therefore of the angles BDA, DBA is double of the angle DAB; wherefore an isosceles triangle ABD is described, having each of the angles at the base double of the third angle. Q. E. F.

PROP. XI. PROB.

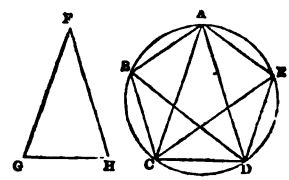
To inscribe an equilateral and equiangular pentagon in a given circle.

Let ABCDE be the given circle: it is required to inscribe in it an equilateral and equiangular pentagon.

Describe (10. 4.) an isosceles triangle FGH, having each of the angles at G, H, double of the angle at F; and in the circle ABCDE inscribe (2. 4.) the triangle ACD equiangular to the triangle FGH, so that the angle CAD be equal to the angle at F, and each of the angles ACD, CDA equal to the angle at G or H, wherefore each of the angles ACD, CDA is double of the angle CAD. Bisect (9. 1.) the angles ACD,

CDA by the straight lines CE, DB, and draw AB, BC, DE, EA: ABCDE is the pentagon required.

Because the angles ACD, CDA are, each, double of CAD, and are bisected by the straight lines CE, DB, the five angles DAC, ACE, ECD, CDB, BDA are equal to one another; but equal angles stand upon equal (26. 3.) circumferences; therefore the five circumferences AB, BC, CD, DE, EA are equal to one another: and equal circumferences are subtended

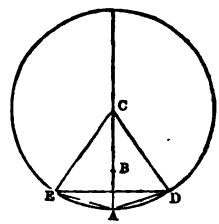


by equal (29. 3.) straight lines; therefore the five straight lines AB, BC, CD, DE, EA are equal to one another; wherefore the pentagon ABCDE is equilateral. It is also equiangular: for since the circumference AB is equal to the circumference DE, if to each be added BCD, the whole ABCD is equal to the whole EDCB; and the angle AED stands on the circumference ABCD, and the angle BAE on the circumference EDCB; therefore the angle BAE is equal (27. 3.) to the angle AED: for the same reason, each of the angles ABC, BCD, CDE is equal to the angle BAE or AED: therefore the pentagon ABCDE is equiangular; and it has been shown that it is equilateral. Wherefore in the given circle, an equilateral and equiangular pentagon has been inscribed. Q. E. F.

Otherwise,

Divide the radius CA of the given circle (11. 2.) so that the rectangle CA, AB may be equivalent to the square of CB; and in the circle place the straight lines AE, AD (1. 4.) each equal to CB, which is not greater than the diameter; join D and E: the line DE is the side of an equilateral and equiangular pentagon inscribed in the circle.

Join C, D and C, E; then by construction, ACD is an isosceles triangle having the angle at C equal to half of either of the angles at the base (10. 4.). Therefore the angle ACD is one fifth of the sum of all the angles of the triangle ACD, or one fifth of two right angles. But the straight line AE is equal to AD by construction; therefore the triangle ACE is equal in every respect to the triangle ACD, and the angle ACE (8. 1.) is equal to ACD. Hence, the angle ECD is equal to two fifths of two right angles, or to one fifth of four right angles, and the arc EAD is one fifth of the circumference of the circle; therefore, if five lines, each equal to the straight line ED, were successively inscribed in the whole circle, there would be formed a pentagon. The equality of the sides is manifest from the construction, and the proof that the angles are equal is the same as that which is given in the text.

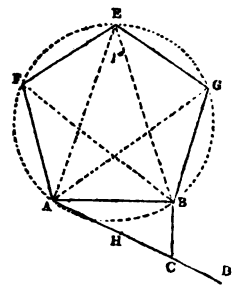


Cor. Since the angle ACD is one fifth of two right angles it is one tenth of four right angles, or of all the angles about C, and the arc AD is one tenth of the circumference of the circle. Therefore, if ten lines, each equal to the straight line AD, were successively inscribed in the whole circle, there would be formed an equilateral and equiangular decagon.

PROP. A. PROB.

Upon a given line as a side, to construct an equilateral and equiangular pentagon.

Let AB be the given side. At the point B draw (11. 1.) the line BC perpendicular to AB and make it equal to half that side; join A, C, and produce the line, and make CD, the part produced, equal to BC or to the half of AB: then upon AB as a base describe (22. 1.) the isosceles triangle AEB having each of the equal sides AE, BE equal to AD. Again, upon these equal sides as bases, describe the isosceles triangles AFE, BGE having each of the sides equal to AB: the figure ABGEF thus formed upon AB is an equilateral and equiangular pentagon.



For let CH be made equal to CD; then DH will be equal to AB, and (11. 2.) the rectangle DA, AH will be equivalent to the square of HD; that is, to the square of AB. Again, since AE and BE are each equal to AD and stand upon AB as a base, the triangle EAB has each of the angles at the base (10. 4.) equal to double the angle at E. Now, if a circle be supposed to be described about the triangle EAB (5. 4.) and the angles EAB, EBA to be bisected by lines drawn from A and B to meet the circle; then the angle AEB and the four which are thus formed at A and B will all be equal to one another; and straight lines joining the points E, A and B with the two points in which the bisecting lines meet the circle will all be equal to one another and to AB (11. 4.). But EG, GB, EF, AF are, by construction, equal to AB; therefore F and G coincide with the points in which the bisecting lines meet the circumference of the circle, and consequently the figure ABGEF is an equilateral and equiangular pentagon. Q. E. F.

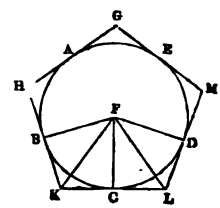
PROP. XII. PROB.

To describe an equilateral and equiangular pentagon about a given circle.

Let ABCDE be the given circle: it is required to describe about it an equilateral and equiangular pentagon.

Let the angles of a pentagon inscribed in the circle, by

the last proposition, be in the points A, B, C, D, E, so that the circumferences AB, BC, CD, DE, EA are equal (11. 4.); and through the points A, B, C, D, E, draw GH, HK, KL, LM, MG, touching (17. 3.) the circle; take the centre F, and join FB, FK, FC, FL, FD. And because the straight line KL touches the circle ABCDE in the point C, to which FC is drawn from the centre F, FC is perpendicular (18. 3.) to KL; therefore each of the angles at C is a right angle: for the same reason, the angles at the points B, D are right angles: and because KC, KB, drawn from the same point K, touch the circle, KC is equal to KB (Cor. A. 36. 3.): also because FB is equal to FC, and FK common to the triangles BFK, CFK, also the base BK is equal to the base KC; therefore the angle BFK is equal (8. 1.) to the angle KFC, and the angle BKF to FKC; wherefore the angle BFC is double of the angle KFC, and BKC double of FKC: for the same reason, the angle CFD is double of the angle CFL, and CLD double of CLF. And because the circumference BC is equal to the circumference CD, the angle BFC is equal (27. 3.) to the angle CFD; and BFC is double of the angle KFC, and CFD double of CFL; therefore the angle KFC is equal to the angle CFL; and the right angle FCK is equal to the right angle FCL: therefore, in the two triangles FKC, FLC there are two angles of one equal to two angles of the other, each to each, and the side FC, which is adjacent to the equal angles in each, is common to both; therefore the other sides shall be equal (26. 1.) to the other sides, and the third angle to the third angle; therefore the straight line KC is equal to CL, and the angle FKC to the angle FLC: and because KC is equal to CL, KL is double of KC: in the same manner it may be shown that HK is double of BK: and because BK is equal to KC, as was demonstrated, and that KL is double of KC, and HK double of BK, HK shall be equal to KL. In like manner it may be shown that GH, GM, ML are each of them equal to HK, or LK: therefore the pentagon GHKLM is equilateral. It is also equiangular: for, since the angle FKC is equal to the angle FLC, and that the angle HKL is double of the angle FKC, and KLM double of FLC, as was before demonstrated, the angle HKL is equal to KLM. And in like manner it may be shown that each of the angles KHG, HGM, GML is equal to the angle HKL, or KLM; therefore the five angles GHK, HKL, KLM, LMG, MGH, being equal to one another, the pentagon GHKLM is



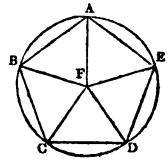
equiangular: and it is equilateral, as was demonstrated; and it is described about the circle ABCDE. Q. E. F.

PROP. XIII. PROB.

To describe a circle about a given equilateral and equiangular pentagon.

Let ABCDE be the given equilateral and equiangular pentagon: it is required to describe a circle about it.

Bisect (9. 1.) the angles BCD, CDE by the straight lines CF, FD, and from the point F, in which they meet, draw the straight lines FB, FA, FE to the points B, A, E. Then, since BC is equal to CD, and CF common to the triangles BCF, DCF, the two sides BC, CF are equal to the two DC, CF; and the angle BCF is equal to the angle DCF; therefore the base BF is equal (4. 1.) to the base FD, and the angle CBF to the angle CDF. And because the angle CDE is double of CDF, and that CDE is equal to CBA, and CDF to CBF, CBA is also double of the angle CBF; therefore the angle ABF is equal to the angle CBF, and the angle ABC is bisected by the straight line BF. In the same manner it may be demonstrated that the angles BAE, AED are bisected by the straight lines AF, FE. Now because the angle BCD is equal to the angle CDE, and that FCD is the half of BCD, and CDF the half of CDE, the angle FCD is equal to FDC; therefore the side CF is equal (6. 1.) to the side FD. In like manner it may be demonstrated that FB, FA, FE are each of them equal to FC, or FD; therefore the five straight lines FA, FB, FC, FD, FE are equal to one another; and the circle described from the centre F, at the distance of one of them, will pass through the extremities of the four others; that is, it will be described about the equilateral and equiangular pentagon ABCDE. Q. E. F.



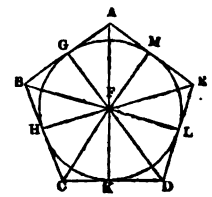
PROP. XIV. PROB.

To inscribe a circle in a given equilateral and equiangular pentagon.

Let ABCDE be the given equilateral and equiangular pentagon: it is required to inscribe in it a circle.

Bisect (9. 1.) the angles BCD, CDE by the straight lines CF, DF, and from the point F, in which they meet, draw the straight lines FB, FA, FE. It may be demonstrated in the

same manner as in the preceding proposition, that the angles CBA, BAE, AED are bisected by those straight lines. From the point F draw (12. 1.) FG, FH, FK, FL, FM perpendiculars to the straight lines AB, BC, CD, DE, EA. And because the angle HCF is equal to KCF, and the right angle FHC equal to the right angle FKC; in the triangles FHC, FKC there are two angles of one equal to two angles of the other, and the side FC, which is opposite to one of the equal angles in each, is common to both; therefore the other sides shall be equal (26. 1.), each to each; wherefore the perpendicular FH is equal to the perpendicular FK. In the same manner it may be demonstrated that FL, FM, FG are each of them equal to FH, or FK; therefore the five straight lines FG, FH, FK, FL, FM are equal to one another. Wherefore the circle described from the centre F, at the distance of one of these five, shall pass through the extremities of the four others, and it shall touch the straight lines AB, BC, CD, DE, EA, because the angles at the points G, H, K, L, M are right angles; and that a straight line drawn from the extremity of the diameter of a circle at right angles to it touches (16. 3.) the circle: therefore each of the straight lines AB, BC, CD, DE, EA touches the circle: wherefore the circle is inscribed in the pentagon ABCDE. Q. E. F.



PROP. XV. PROB.

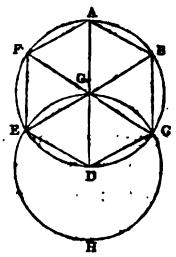
To inscribe an equilateral and equiangular hexagon in a given circle.

Let ABCDEF be the given circle: it is required to inscribe in it an equilateral and equiangular hexagon.

Find the centre G of the circle ABCDEF, and draw the diameter AGD; and from D, as a centre, at the distance DG, describe the circle EGCH; draw EG, CG, and produce them to the points B, F; also draw AB, BC, CD, DE, EF, FA: the hexagon ABCDEF is equilateral and equiangular.

Because G is the centre of the circle ABCDEF, GE is equal to GD; and because D is the centre of the circle EGCH, DE is equal to DG; wherefore GE is equal to ED, and the triangle EGD is equilateral; and therefore its three angles EGD, GDE, DEG are equal to one another, because the angles at the base of an isosceles triangle are equal (5. 1.): and the three angles of a triangle are equal (32. 1.) to two right angles; therefore the angle EGD is the third part of two

right angles. In the same manner it may be demonstrated that the angle DGC is also the third part of two right angles. And because the straight line GC makes with EB the adjacent angles EGC, CGB equal (13. 1.) to two right angles, the remaining angle CGB is the third part of two right angles; therefore the angles EGD, DGC, CGB are equal to one another: and to these are equal (15. 1.) the vertical opposite angles BGA, AGF, FGE: therefore the six angles EGD, DGC, CGB, BGA, AGF, FGE are equal to one another.



But equal angles stand upon equal (26. 3.) circumferences; therefore the six circumferences AB, BC, CD, DE, EF, FA are equal to one another: and equal circumferences are subtended by equal (29. 3.) straight lines; therefore the six straight lines are equal to one another, and the hexagon ABCDEF is equilateral. It is also equiangular; for, since the circumference AF is equal to ED, to each of these add the circumference ABCD; therefore the whole circumference FABCD shall be equal to the whole EDCBA: and the angle FED stands upon the circumference FABCD, and the angle AFE upon EDCBA; therefore the angle AFE is equal to FED. In the same manner it may be demonstrated that the other angles of the hexagon ABCDEF are each of them equal to the angle AFE, or FED: therefore the hexagon is equiangular; and it is equilateral, as was shown; and it is inscribed in the given circle ABCDEF. Q. E. F.

Cor. From this it is manifest that the side of the hexagon is equal to the straight line from the centre, that is, to the semidiameter of the circle.

And if, through the points A, B, C, D, E, F there be drawn straight lines touching the circle, an equilateral and equiangular hexagon shall be described about it, which may be demonstrated from what has been said of the pentagon; and likewise a circle may be inscribed in a given equilateral and equiangular hexagon, and circumscribed about it, by a method like to that used for a pentagon.

N.B. In practice it will evidently suffice to inscribe in the circle a line equal to the semidiameter, and to repeat the inscription within the whole of the circumference.

PROP. B. PROB.

Upon a given straight line as a side, to describe an equilateral and equiangular hexagon.

Let ED, in the above figure, be the given side. Upon it describe an equilateral triangle EGD (1. 1.); upon each of the two sides DG and EG describe the equilateral triangles DGC, EGF; and again, upon GC and GF describe the equilateral triangles GBC, GAF; lastly, join A, B: then ABCDEF is the required hexagon: for all the sides are equal by construction, and all the angles of the polygon are equal to one another, each of them being equal to two thirds of two right angles: wherefore, &c. Q. E. F.

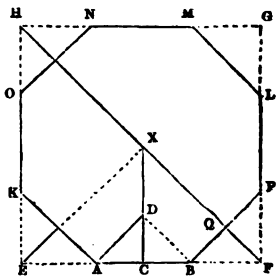
N.B. In practice it would be more accurate to draw a line, as FGC, and on it make FG and GC each equal to the given side: then, if the equilateral triangles FAG, FEG, GDC, GBC were to be described, and the points E, D and A, B respectively, joined, the hexagon would be constructed.

PROP. C. PROB.

UPON a given line as a side, to describe an equilateral and equiangular octagon.

Let AB be the given side. Bisect AB in C (10. 1.), and draw CD (11. 1.) perpendicular to, and equal to the half of AB; on AB, produced both ways, make AE and BF each equal to AD, and on EF construct the square EFGH (46. 1.); again, on the sides of the square, and from the angular points, make EK, FP, GL, &c., each equal to AE, or AD, and join the points as in the figure: ABPLMNOK is the required octagon.

For imagine D and B to be joined: then, by construction, DC is equal to AC, or CB, and the angles at C are right angles; therefore, in the triangles ACD, DCB, AD is equal to DB; and each of the angles DAB, DBA being half a right angle, the angle ADB is a right angle (32. 1.). Next, AE and EK are each equal to AD, or DB, and the angle at E is equal to ADB; that is, to a right angle; therefore, in the triangles AEK, ADB, AK is equal to AB. In like manner it may be proved that BP, ML, and NO are each equal to AB. Again, EH is equal to EF, each being the side of a square, also EK and HO are, by construction, equal to EA, or BF; therefore, taking equals from equals, KO is equal to AB: in like manner, NM and LP are



each equal to AB; therefore the octagon is equilateral. It is also equiangular; for, since the angles EAK, PBF, AKE, &c., are each equal to half a right angle, taking each of them from two right angles, the remainders, KAB, ABP, &c., will each be equal to a right angle and half a right angle; that is, they will be equal to one another: wherefore, &c. Q. E. F.

N.B. It may be proved that AF, BB, &c. are each equal to half the diagonal of the square EFGH. Draw the diagonal HF, and produce CD to meet it in X, also join E, X; then, since AC is equal to CB, and AB to BF, BC is equal to CF; and in the triangles ECX, FCX, EC is equal to FC, CX is common, and the angles at C are right angles; therefore EX is equal to FX. In like manner it may be proved that EX is equal to HX, &c.; therefore X is the centre of a circle circumscribing the square, and FX, or HX is equal to half the diagonal of the square.

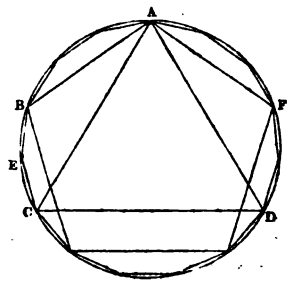
Now, since each of the angles FBQ, BFQ is half a right angle, each of them is equal to the angle CBD, or CDB, and, by construction, BF is equal to BD, or AD; therefore BQ and FQ are each equal to CB, or AC, and the angle at Q is equal to the angle at C, that is, to a right angle: and in the triangles BXQ, BXC, BQ is equal to BC, BX is common, and the angles at Q and C are right angles; therefore XQ is equal to XC: but XC is equal (6. 1.) to CF, because the angle CXF is equal to CFX, each of them being half a right angle; therefore XQ is equal to CF: then, since FQ is equal to AC, adding equals to equals, XF is equal to AF; that is, AF is equal to half the diagonal of the square. In like manner it may be shown that BB, EO, &c., are, each, equal to the half diagonal.

PROP. XVI. PROB.

To inscribe an equilateral and equiangular quindecagon in a given circle.

Let ABCD be the given circle: it is required to inscribe in it an equilateral and equiangular quindecagon.

Let AC be the side of an equilateral triangle inscribed (2. 4.) in the circle, and AB the side of an equilateral and equiangular pentagon inscribed (11. 4.) in the same; therefore, of such equal parts as the whole circumference ABCDF contains fifteen, the circumference ABC, being the third part of the whole, contains five; and the circumference AB, which is the fifth part of the whole, contains three; therefore BC, their difference, contains two of the same parts. Bisect (30. 3.) BC in E; therefore BE, EC are, each of them, the fifteenth part of the whole circumference ABCD: therefore if the straight



lines BE, EC be drawn, and straight lines equal to them be placed round (1. 4.) in the whole circle, an equilateral and equiangular quindecagon shall be inscribed in it. Q. E. F.

And, in the same manner as was done in the pentagon, if, through the points of division made by inscribing the quindecagon, straight lines be drawn touching the circle, an equilateral and equiangular quindecagon shall be described about it: and likewise, as in the pentagon, a circle may be inscribed in a given equilateral and equiangular quindecagon, and circumscribed about it.

PROP. D. PROB.

To inscribe in a circle, equilateral and equiangular polygons, having any number whatever of sides in the series of numbers eight, sixteen, thirty-two, &c.

First, inscribe in the circle a square (6. 4.); then bisect (30. 3.) the arc of the segment cut off from the whole circumference by each side of the square, and join the points of bisection to the angular points of the square by straight lines: the figure thus formed is an equilateral and equiangular octagon. Again, bisect the arc cut off by each side of the octagon and join the points of bisection to the angular points of the octagon by straight lines: the figure thus formed is an equilateral and equiangular polygon of sixteen sides. By proceeding continually in this manner, a polygon having any number of sides in the above series of numbers may be inscribed in the circle.

Such polygons are equilateral, because equal portions of the circumference are continually bisected; therefore (29. 3.) the straight lines which join the points are equal to one another: and they may be proved to be equiangular, as in Props. 11. and 15. of this book: wherefore, &c. Q. E. F.

In like manner, inscribing in a circle a pentagon, and bisecting the segments continually, there may be inscribed in the same circle polygons having ten, twenty, forty, &c. sides.

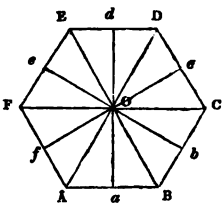
By drawing lines touching the circle at all the angular points of any inscribed polygon, as in Prop. 12., or at points in the circumference, at the middle of the arcs joining those angular points, polygons of an equal number of sides may be described about a circle.

N.B. Polygons which are equilateral and equiangular, are frequently called regular polygons.

PROP. E. THEOR.

An equilateral and equiangular polygon is equivalent to the rectangle contained by a line let fall from the centre of a circle described about the polygon, perpendicularly upon any one of the sides, and another line which is equal to half the perimeter of the polygon.

Let ABCDEF represent any such polygon, and let O be the centre of a circle described about it. Draw OA, OB, OC, &c. from the centre of the circle to the angular points of the figure: these will be radii of the circle, and consequently equal to one another. Let fall O*a*, O*b*, &c., from the centre O perpendicularly on the several sides of the polygon: then because AB, BC, &c. are equal lines inscribed in a circle, they are equally distant from the centre (14. 3.); therefore O*a*, O*b*, &c. are equal to one another: these also bisect the lines AB, BC, &c. (3. 3.) Now the triangle AOB is equivalent to the rectangle O*a*, *a*B; that is, to the rectangle of the perpendicular and half the side AB (42. 1.); and the like is true of the other triangles; therefore the sum of all the triangles, that is, the polygon ABCDEF, is equivalent to the rectangle of the perpendicular and half the sum of all the sides; or to the rectangle of the perpendicular and half the perimeter of the polygon. Q. E. D.



ELEMENTS

OF

GEOMETRY.

PROPORTION.

N. B. *This Tract on Proportion is inserted in the place of Euclid's Fifth Book.*

DEFINITIONS.

I.

HOMOGENEOUS magnitudes are such as can be added to, and subtracted from one another.

II.

A greater magnitude, or number, is said to be a multiple of a less magnitude, or number, when the greater is measured by the less, that is, when the greater contains the less a certain number of times exactly.

III.

A less magnitude, or number, is said to be a part, or a submultiple, of a greater magnitude, or number, when the less measures the greater, that is when the less is contained in the greater a certain number of times exactly.

IV.

Equimultiples of magnitudes are such as contain those magnitudes the same number of times; and like parts are such as are contained in their wholes the same number of times.

V.

Magnitudes are commensurable which have a common measure; that is, which contain one and the same magnitude any number of times exactly.

VI.

Magnitudes are incommensurable which have no common measure.

VII.

Ratio is the relation which one magnitude has to another of the same kind, with respect to quantity; or when the first is considered as composed of parts of the other.

VIII.

The first magnitude, or that which is compared to the other, is called the antecedent: that with which it is compared is called the consequent.

IX.

A ratio of equality is that in which the antecedent is equal to the consequent; otherwise, it is called a ratio of inequality: of majority when the antecedent is greater than the consequent; of minority when the antecedent is less than the consequent.

X.

When the first of four magnitudes or quantities contains the second, or a part of the second (without a remainder) just as often as the third contains the fourth, or a like part of the fourth, the first is said to have to the second, the same ratio which the third has to the fourth; or, the ratio of the first to the second is said to be equal to the ratio of the third to the fourth.

Definition of proportion according to Euclid, (Def. V., Book V.)

"The first of four magnitudes is said to have the same ratio "to the second, which the third has to the fourth, when any "equimultiples whatsoever of the first and third being "taken, and any equimultiples whatsoever of the second and "fourth; if the multiple of the first be equal to, greater "than, or less than the multiple of the second, the multiple "of the third is also equal to, greater than, or less than that "of the fourth."

XI.

When the first has to the second, the same ratio which the third has to the fourth, the four magnitudes make a proportion or analogy; and they are said to be proportional.

The four magnitudes or quantities are called the terms of the proportion.

When four magnitudes, or quantities, A, B, C, D are proportional, it is usual to say, that A is to B as C to D, and to write them thus, A : B : : C : D.

XII.

The first and fourth magnitudes of a proportion are called the extreme terms; the two intermediate magnitudes are called the mean terms.

XIII.

The homologous terms of a proportion are the antecedents, and the consequents of the ratios.

XIV.

In any number of magnitudes greater than two, when the first has to the second, the same ratio that the second has to the third, and the second to the third, the same ratio that the third has to the fourth, and so on, the magnitudes are said to be continual proportionals, or to be in continued proportion.

Continual proportionals are usually thus written, viz. A : B : : B : C : : C : D : : D : E.

XV.

When three magnitudes are continual proportionals, the second is said to be a mean proportional between the two extreme terms.

XVI.

In any number of magnitudes of the same kind, the ratio of the first to the last is said to be compounded of the ratios of the first to the second, of the second to the third, of the third to the fourth, and so on to the last magnitude.

XVII.

If three magnitudes be continual proportionals, the ratio of the first to the third is said to be duplicate of the ratio of the first to the second.

XVIII.

If four magnitudes be continual proportionals, the ratio of the first to the fourth is said to be triplicate of the ratio of the first to the second.

A duplicate, or triplicate, ratio is nothing more than a ratio compounded of two, or three, equal ratios.

COROLLARIES FROM THE DEFINITIONS.

I.

In every proportion, the ratios of the first to the second, and of the third to the fourth, are both, ratios of majority; or both, of equality; or both, of minority.

II.

And hence, in every proportion, if the first term be greater than the second, the third is also greater than the fourth; if equal, equal; and if less, less.

III.

Equal magnitudes have the same ratio to the same magnitude; and the same magnitude has the same ratio to equal magnitudes.

IV.

Magnitudes which have the same ratio to the same magnitude are equal to one another; and those to which the same magnitude has the same ratio are equal to one another.

V.

Ratios which are equal to the same ratio, or to equal ratios, are equal to one another.

AXIOMS.

I.

A magnitude which measures two other magnitudes, will likewise measure both their sum and their difference.

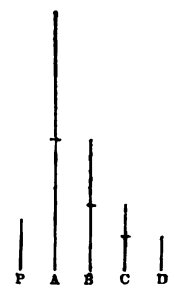
II.

In any proportion among magnitudes, the first and third terms remaining the same, if the second be increased or diminished by any magnitude, the fourth (the proportion still subsisting) will be increased or diminished by some magnitude.

PROP. A.

If from the greater of two unequal magnitudes there be taken its half, and from the remainder its half, and so on: there shall at length remain a magnitude less than the least of the proposed magnitudes, how small soever it may be.

Let A, B, C, D, &c. be a series of magnitudes such that B is half of A, C half of B, D half of C, and so on; and let P be any magnitude how small soever: the magnitudes of the series, if continued far enough, will at length become less than P.



For P will be contained in A a certain number of times, either exactly, or with a remainder less than itself. But B is contained in A twice; and C is contained in it four times; and D eight times, and so on. Wherefore, by continuing the series, we shall at length come to a magnitude which is contained in A more often than P is: that magnitude and every one following it in the series is therefore less than P.

Much more is this proposition true, when every magnitude in the series is less than half of that which goes before it: that is, when the part which being taken from A would leave B, is greater than the half of A; the part which being taken from B would leave C, is greater than the half of B; and so on. Hence, by taking from A its half, or more than its half; from the remainder its half, or more than its half; and so on, there will remain at length a magnitude less than the least that can be assigned.

Cor. 1. By bisecting any magnitude, or dividing it into two equal parts; then bisecting each of these equal parts; again bisecting each of the latter parts, and so on; the magnitude will at length be divided into equal parts each of which is equal to, or less than, the least magnitude that can be assigned.

Cor. 2. If a magnitude be less than any that can be assigned, a magnitude which is a given multiple of it is also less than any that can be assigned.

For, let there be two magnitudes, one of which is a given multiple of the other, and from each let there be taken its half, from the remainder its half, and so on: that which remains of the greater magnitude will always be the same

given multiple of that which remains of the other. But while that which remains of the greater magnitude, is equal to any magnitude that can be assigned, that which remains of the other, being a given part of it, is also equal to a magnitude which may be assigned; therefore the remainders of both the original magnitudes become at the same time (that is, after an equal number of bisections of the two magnitudes) less than any magnitude that can be assigned. Hence, if a magnitude, &c.

From this Cor. and from Def. 10. it follows that when four quantities are proportionals, of which the first and second terms do not exceed any assignable magnitudes; if the third term be less than any magnitude that can be assigned, the fourth term will also be less than any magnitude that can be assigned: much more also will the fourth term be less than any assignable magnitude, if only the first term is a given magnitude, and the three other terms are less than any that can be assigned.

Cor. 3. If a line be less than any that can be assigned, its square will be less than any surface that can be assigned.

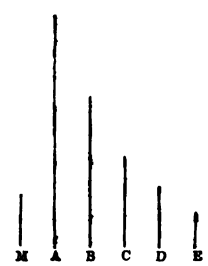
Cor. 4. A rectangle contained by two lines, one of which is given, and the other is less than any line that can be assigned, is a magnitude less than any surface that can be assigned: for it is that which remains when from a given rectangle there is taken one which (having one side equal to the given line) is equal to its half, or more than its half; from the remainder, one which (having one side equal to the given line) is equal to half, or more than half, of that remainder; and so on. Hence, two rectangles having the same, or equal altitudes, and whose bases differ from one another by a line less than any that can be assigned, will differ from one another by a surface less than any that can be assigned. Again, a rectangle contained by two lines which differ from two others by lines less than any that can be assigned, will differ from the rectangle contained by the two other lines by a surface less than any that can be assigned. Also if the difference between two lines is less than any line that can be assigned, the difference between the squares of those lines will be less than any surface that can be assigned.

PROP. B.

Two unequal homogeneous magnitudes being given, if the less be taken from the greater, as often as it can,

and in like manner, what remains be taken from the less, and so on continually, by alternate subtractions; and, if none of the remainders measures that which goes before it: the two quantities are incommensurable.

Let A and B be two unequal magnitudes of the same kind, A being greater than B: take B out of A till there remains C less than B; take C out of B till there remains D less than C; and again D out of C till there remains E less than D; and supposing that no remainder E measures the preceding one D, conceive the same operations to be continued indefinitely: A and B are incommensurable.



If A and B be not incommensurable, let M measure them both. And because M measures B, it will measure its multiple, which is contained in A; but it likewise measures A; wherefore it will measure C, the difference between A and the multiple of B (1. Axiom, Proportion). Again, because M measures C, it will measure its multiple, which is contained in B: but it likewise measures B; wherefore it will measure D, the difference between B and the multiple of C. In like manner it is proved that M will measure E, and all the succeeding magnitudes in the series of remainders.

Now because A contains B at least once with the remainder C less than B, C is less than half of A. And because C contains D at least once with the remainder E less than D, E is less than half of C, and so on. The alternate magnitudes A, C, E, &c., therefore form a series in which every succeeding magnitude is less than half the preceding one, and being continued far enough, they will become less than M (A. Proportion). Suppose that in the alternate magnitudes A, C, E, &c., one of them, E, is at length found, which is less than M; then, because the remainders C, D, E, &c., continually decrease, all those that follow E will likewise be less than M: but it has been proved that M measures all the remainders however far continued; which is absurd, since a magnitude cannot measure that which is less than itself. Wherefore, A and B cannot, both, be measured by the same magnitude, that is, they are incommensurable.

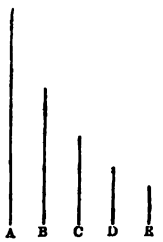
PROP. C.

Two homogeneous magnitudes or quantities, which are commensurable, being given, it is required to find their greatest common measure.

Let A and B be two commensurable magnitudes, it is required to find their greatest common measure.

Supposing A to be greater than B, take B from A as often as it can, and let the remainder be C; take C from B as often as it can, and let the remainder be D; continue the same operations till a remainder E be found that exactly measures the magnitude D, immediately preceding it: then E is the greatest common measure required.

In the first place, the series of remainders must at length stop, as is here supposed; for if it did not, the magnitudes A and B would be incommensurable (B. Pr.) contrary to the hypothesis. And next, no magnitude greater than E can measure both A and B; for it has already been proved that any common measure of A and B must measure all the successive remainders C, D, E (B. Pr.), &c., and E can have no measure greater than itself.



Lastly, E will measure both A and B. For since E measures D, it will measure any multiple of D. But C is composed of a multiple of D, and of E itself; wherefore E will measure C. Consequently E will measure any multiple of C; and it likewise measures D; wherefore it will measure B, which is equal to a multiple of C together with D. And in like manner it is proved that E will measure A, which is equal to a multiple of B together with C. Wherefore E is a common measure of A and B, and it is their greatest common measure. Q. E. I.

Cor. And since it is proved that every common measure of A and B will measure all the successive remainders C, D, E (B. Pr.); it is manifest that no magnitude or quantity which does not measure E, can be a common measure of A and B. Conversely, any magnitude which measures E, will be a common measure of A and B.

And hence the greatest common measure of three magnitudes may be found; namely, by finding the greatest common measure of two of them, and then, again, taking the greatest

common measure of the third, and the common measure already found. And the same method will apply to any number of magnitudes or quantities.

SCHOLIUM. It is demonstrated in Euclid's tenth book (Prop. 5.) that, when magnitudes are commensurable, the relation between any two, with respect to quantity or extent, may be the same as that of one number to another; and this is sufficiently manifest: for if, of two magnitudes A and B of the same kind (as two straight lines) having a common measure, A be capable of being divided into a parts, each equal to that common measure, and B into b such parts (a and b representing some integral numbers); the ratio of A to B is indicated by the circumstance that A contains a certain number a, of the b parts into which B was divided, and is expressed by the relation that a bears to b, or by the numerical fraction $\frac{a}{b}$. Now let C and D be other magnitudes of a like kind, having also a common measure, and let it be supposed that C contains c parts, and D contains d parts, each equal to that common measure; then the ratio of C to D will be indicated by the circumstance that C contains a certain number c, of the d parts into which D was divided, and will be expressed by the relation that c bears to d, or by the numerical fraction $\frac{c}{d}$: it will follow that if the ratio of A to B be equal to that of C to D, so that A : B :: C : D, that equality may be expressed by writing $a : b :: c : d$, or by making $\frac{a}{b}$ equal to $\frac{c}{d}$. Thus, for a proportion between four geometrical magnitudes A, B, C, D, or between four numbers a, b, c, d, there may be substituted the equivalent numerical equation

$$\frac{a}{b} = \frac{c}{d},$$

where the symbol $=$ is employed to denote equality.

N. B. Besides this symbol, there will hereafter be occasionally employed the symbols $+$ and $-$; one of which coming between two magnitudes or quantities denotes that the quantity following it is added to, or subtracted from, that which precedes it. Also, when two letters, as a, b, expressing numbers, are placed together, as ab, the term is to be understood as signifying that the number a is multiplied by the number b: again, a^2 and a^3 denote respectively the product of a by a, or the square of a, and the continued product of a, a, a, or the cube of a.

PROP. D.

Two incommensurable magnitudes being given, a part of one of them may be found which shall measure the other when that other is increased or diminished by a magnitude less than the least that can be assigned.

Let A and B be two incommensurable magnitudes: it is required to obtain, approximatively, a common measure of those magnitudes. Suppose either of the two magnitudes, as A, to be divided into two equal parts; each of these into two equal parts, and so on; the given magnitude will at length (Prop. A.) be divided into equal parts, each of which is either equal to, or less than, the least magnitude that can be assigned. Then, A and B being incommensurable, one of these parts will be contained in the other magnitude B, a certain number of times with a remainder (Prop. B.); that remainder will be less than the part of A, and consequently less than the least magnitude that can be assigned. Therefore, the part of A will measure a magnitude which is less than B by a magnitude less than the least that can be assigned. Again, if the part of A be applied to B one time more, the magnitude thus made up of parts of A will exceed B, and the excess will be less than the part of A; that is, less than the least magnitude that can be assigned: therefore the part of A will measure a magnitude which exceeds B by a magnitude less than the least that can be assigned. Hence a part of either of the magnitudes A and B can, by continual bisections of that magnitude, at length be found which may be considered as a common measure of the two magnitudes; and the degree of approximation will depend upon the number of bisections.

SCHOLIUM. If the incommensurable magnitudes A and B be represented by two numbers, it will follow from this proposition that a quantity may be conceived to exist which shall measure the two numbers when one of them is increased or diminished by a quantity less than the least that can be assigned. Therefore the theorems relating to proportions among numbers which are incommensurable may be considered as proved by the same processes as are used when the numbers are commensurable.

FIRST GENERAL THEOREM.

Let there be two geometrical magnitudes A and B, whose equality to each other cannot be directly proved: then if there be found two magnitudes $A_,$ and A' which differ from A, the one in defect, and the other in excess, by magnitudes less than any that can be assigned, and which also differ from B in defect and excess by magnitudes less than any that can be assigned; it will follow that B may be considered as equal to A.

For, either B will be precisely equal to A, or it will differ from it by a magnitude less than either of those by which it differs from $A_,$ or A'; and either of these last differences is by hypothesis less than any assignable magnitude; therefore the difference (if there be any) between A and B must be less than any assignable magnitude. Thus it may be considered that B is equal to A. Q. E. D.

SECOND GENERAL THEOREM.

Let $A_,$ A, A' be geometrical magnitudes of one kind (surfaces, for example), such that the excess of A above $A_,$ and of A' above A, are each less than any assignable magnitude; and let $B_,$ B, B', be geometrical magnitudes of the same, or of another kind (lines, for example), such that the excess of B above $B_,$ and of B' above B are each less than any assignable magnitude. Also, from given proportions between commensurable magnitudes,

let $m : A_, :: n : B_,$

and $m : A' :: n : B'$;

where m and n are magnitudes of the same nature respectively as A and B, and where n and B are incommensurable: it is required to prove that $m : A :: n : B$.

In the two preceding proportions, if $A_,$ be increased, $B_,$ will increase (2. Ax. Pr.), and if A' be diminished, B' will diminish; therefore if $A_,$ be increased, so as to become equal to A, which is less than A', $B_,$ will increase to an extent less than B'. But B is greater than $B_,$ and less than B', and the difference between $B_,$ and B' is equal to, or less than, the least magnitude that can be assigned; therefore, when the second term in either of those proportions becomes equal to A, the fourth term will either be accurately equal to B, or it will differ from it by a magnitude less than the least that can be assigned. Thus it may be considered that $m : A :: n : B$. Q. E. D.

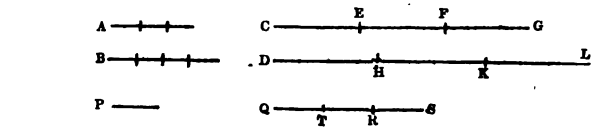
PROP. I.

MAGNITUDES have the same ratio to one another which their equimultiples have.

Let A and B be two magnitudes, and CG and DL equimultiples of them:

A : B :: CG : DL.

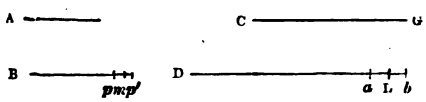
Let P be a common measure of A and B, and take QS the same multiple of P that CG is of A, and DL of B: and let



CE, EF, FG be each equal to A; DH, HK, KL each equal to E; and QT, TR, RS each equal to P. Then CE will contain QT, and EF will contain TR, and FG will contain RS, as often as A contains P; wherefore the whole CG will contain the whole QS, as often as A contains P. In like manner, DH will contain QT, and HK will contain TR, and KL will contain RS, as often as B contains P; wherefore the whole DL will contain the whole QS as often as B contains P. Wherefore, P and QS are like parts of B and DL, and it has been proved that they are contained the same number of times in A and CG; but four magnitudes are proportional when the first contains a part of the second as often as the third contains a like part of the fourth; wherefore A, B, CG, DL are proportional (10 Def. Pr.), that is,

A : B :: CG : DL.

If the magnitudes A and B*m* are incommensurable, by continual bisections of one of them, as A, there may be at length found (D. Pr.), a part which, being equal to, or less than, the least magnitude that can be assigned, will be contained in B*m* a certain number of times with a remainder, and one time more with an excess, less than any magnitude that can be assigned. Let *pm*, *mp'* be such defect and excess respectively; and take the same multiple of the part of A that CG is of A or DL of B*m*: then because CG and DL are equimultiples of A and of B*m*, this multiple of the part of A will be contained in CG exactly as many times as the part is contained in A; and if D*a*, D*b* be the same mul-



tiples of Bp and of Bp' respectively, that CG is of A, the multiple of the part of A will be contained in Da and Db exactly as many times as the part is contained in Bp and Bp'. But aL and Lb are the same multiples of pm and of mp' that CG is of A, or DL is of Bm; and pm, mp' are each less than any assignable magnitude; therefore, (2. Cor. A. Pr.) aL, Lb are each also less than any assignable magnitude.

Now, by the former part of the proposition

A : Bp :: CG : Da

and A : Bp' :: CG : Db;

therefore, by the second General Theorem,

A : Bm :: CG : DL.

N. B. On the like principle all the following propositions may be demonstrated when the magnitudes are incommensurable.

Algebraically.

Let the ratio of the two magnitudes A and B, or of A and Bm be expressed by that of a to b, or by the fraction $\frac{a}{b}$, where a and b denote numbers: and let the equimultiples of A and B or of A and Bm be represented by na and nb in numbers; then the ratio between these multiples may be expressed by $\frac{na}{nb}$.

But, in numbers, $\frac{a}{b} = \frac{na}{nb}$; therefore (Schol. C.)

$$a : b :: na : nb,$$

and this represents the proportion A : B :: CG : DL. That is, magnitudes or quantities have the same ratio to one another that their equimultiples have.

Cor. Magnitudes have the same ratio to one another which their like parts have. For wholes are equimultiples of their like parts.

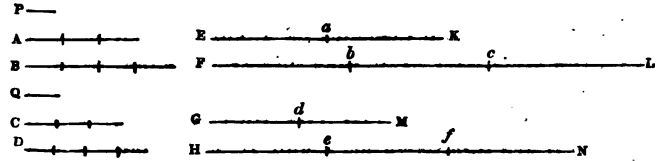
PROP. II.

In a proportion, if there be taken any equimultiples of the antecedents, and also any equimultiples of the consequents: the four multiples shall be proportional.

Let A : B :: C : D; and let EK and GM be any equimultiples of A and C, and FL and HN any equimultiples of

B and D: the four quantities EK, FL, GM, and HN are proportionals.

Let E*a*, *a*K be each equal to A; F*b*, *bc*, *c*L each equal to B; G*d*, *d*M each equal to C; H*e*, *ef*, *f*N each equal to D.



Take P, a common measure of A and B, and let D contain Q as often as B contains P; then because A, B, C, D are proportional, C will contain Q as often as A contains P (10. Def. Pr.). And because there are, in EK, as many parts E*a*, *a*K, each equal to A, as there are, in GM, parts G*d*, *d*M, each equal to C; and because each of the former parts E*a*, *a*K contains P as many times as each of the latter parts G*d*, *d*M contains Q; therefore the whole EK will contain P as often as the whole GM contains Q. And, in like manner, because the parts F*b*, *bc*, *c*L are as many in number as the parts H*e*, *ef*, *f*N, and because each of the former parts, F*b*, *bc*, *c*L contains P as often as each of the latter parts H*e*, *ef*, *f*N contains Q; therefore the whole FL will contain P as often as the whole HN contains Q. The quantities P and Q are, therefore, like parts of FL and HN, and it has been shown that they are contained the same number of times in EK and GM; wherefore EK, FL, GM, HN are proportional (10. Def. Pr.), that is,

EK : FL :: GM : HN.

Algebraically. Let the proportion between the four magnitudes A, B, C, D be represented by $a : b :: c : d$ in numbers: then, if mA, mC be equimultiples of A and C, and nB, nD be equimultiples of B and D, the proportion mA : nB, :: mC : nD, which it is required to prove, may be represented by

$$ma : nb :: mc : nd.$$

Now, since $a : b :: c : d$, it follows (Schol. C.) that $\frac{a}{b} = \frac{c}{d}$.

Multiply each of these equals by the numerical fraction $\frac{m}{n}$; then $\frac{ma}{nb} = \frac{mc}{nd}$, and (Schol. C.) $ma : nb :: mc : nd$, which represents the proportion EK : FL :: GM : HN.

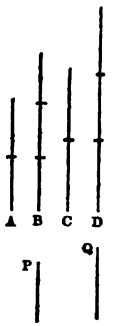
PROP. III.

If four magnitudes be proportional, they are also pro-

portional when taken inversely: that is, the second is to the first as the fourth is to the third.

Let A : B :: C : D, then, by inversion, B : A :: D : C.

Let P be a common measure of A and B, and let Q be the same part of D that P is of B; then, because A, B, C, D are proportional, C will contain Q as often as A contains P (10. Def. Pr.). Wherefore P and Q are like parts of A and C, and they are contained the same number of times in B and D; wherefore B, A, D, C are proportional (10. Def. Pr.), that is,



B : A :: D : C.

Algebraically. Let the proportion A : B :: C : D be represented by $a : b :: c : d$ in numbers: then $b : a :: d : c$.

For (Schol. C.) $\frac{a}{b} = \frac{c}{d}$; and hence, in numbers, (dividing unity by each of these equals) $\frac{b}{a} = \frac{d}{c}$ (or Schol. C.) $b : a :: d : c$; and this represents the proportion

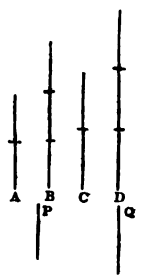
B : A :: D : C.

PROP. IV.

If four magnitudes of the same kind be proportional, they will also be proportional when taken alternately: that is, the first will be to the third as the second to the fourth.

Let A : B :: C : D, and let A, B, C, D be magnitudes of the same kind: then, by alternation, A : C :: B : D.

Let P be a common measure of A and B, and let Q be the same part of D that P is of B; and because A, B, C, D are proportional, Q will be contained in C as often as P is contained in A (10. Def. Pr.). Wherefore A and C are equimultiples of P and Q; and likewise B and D are equimultiples of P and Q. Consequently,



P : Q :: A : C (1. Pr.)

and P : Q :: B : D;

wherefore, by equality of ratios,

A : C :: B : D. (5 Cor. Def. Pr.)

Algebraically. Let the proportion A : B :: C : D be represented by the proportion $a : b :: c : d$ in numbers; then

$$a : c :: b : d.$$

For, since $a : b :: c \cdot d$, $\frac{a}{b} = \frac{c}{d}$ (Schol. prop. C.); multiply these equal quantities by b; then $a = \frac{bc}{d}$. Again, divide these equals by c; then $\frac{a}{c} = \frac{b}{d}$. Consequently, (Schol. C.) $a : c :: b : d$; and this represents the proportion A : C :: B : D.

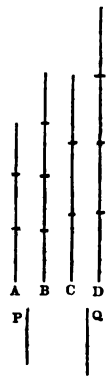
PROP. V.

If four magnitudes be proportional, they will also be proportional by composition: that is, the sum of the first and second will be to the second, as the sum of the third and fourth is to the fourth.

Let A : B :: C : D, then, by composition,

A + B : B :: C + D : D.

Let P measure both A and B, and let Q be the same part of D that P is of B; then will C contain Q as often as A contains P (10. Def. Pr.) And because A contains P as often as C contains Q; and that B contains P as often as D contains Q; therefore it is plain that the sum A + B will contain P, as often as the sum C + D contains Q. Wherefore, because P and Q are like parts of B and D, A + B : B :: C + D : D (10. Def. Pr.)



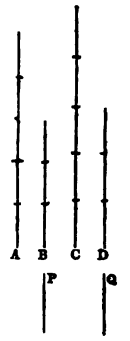
Algebraically. Let the proportion A : B :: C : D be represented by $a : b :: c : d$; then, (Schol. C.) $\frac{a}{b} = \frac{c}{d}$. But, in numbers, $\frac{a}{b} + 1 = \frac{c}{d} + 1$ or $\frac{a+b}{b} = \frac{c+d}{d}$; therefore $a + b : b :: c + d : d$, which represents the proportion A + B : B :: C + D : D.

PROP. VI.

If four magnitudes be proportional, they will also be proportional by division: that is, the difference of the first and second will be to the second as the difference of the third and fourth is to the fourth.

Let A : B :: C : D, and let A be greater than B, and consequently C greater than D; then, by division,

A — B : B :: C — D : D.



Let P measure both A and B, and let Q be the same part of D that P is of B; then will Q be contained in C the same number of times that P is contained in A (10. Def. Pr.). And because A contains P as often as C contains Q; and that B contains P as often as D contains Q; therefore the difference A — B will contain P as often as the difference C — D contains Q. Wherefore, because P and Q are like parts of B and D,

A — B : B :: C — D : D (10. Def. Pr.).

The demonstration is entirely similar when B is greater than A, and D greater than C.

Algebraically. Let the proportion A : B :: C : D be represented by $a : b :: c : d$ in numbers; then (Schol. C.) $\frac{a}{b} = \frac{c}{d}$. But $\frac{a}{b} - 1 = \frac{c}{d} - 1$, or $\frac{a-b}{b} = \frac{c-d}{d}$; therefore $a - b : b :: c - d : d$, which represents

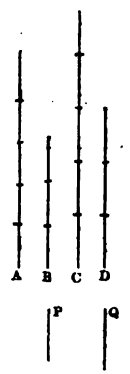
A — B : B :: C — D : D.

PROP. VII.

If four magnitudes be proportional, they will also be proportional by conversion: that is, the first will be to the difference between the first and second as the third is to the difference between the third and fourth.

Let A : B :: C : D; and let A be greater than B, and consequently C greater than D; then, by conversion,

A : A — B :: C : C — D.



Let P measure both A and B, and take Q the same part of D that P is of B; then will C contain Q as often as A contains P. Because A contains P as often as C contains Q; and that B contains P as often as D contains Q; therefore the difference A — B will contain P as often as the difference C — D contains Q. Consequently P and Q are like parts of A — B and C — D; and they are contained the same number of times in A and C; wherefore,

A : A — B :: C : C — D (10. Def. Pr.)

The demonstration is entirely similar when B is greater than A, and D greater than C.

COR. It may be proved in the same manner that

A : A + B :: C : C + D.

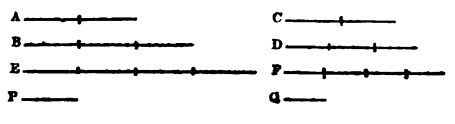
Algebraically. Let the proportion A : B :: C : D be represented by $a : b :: c : d$ in numbers; then (3. Pr.) $b : a :: d : c$ and (Schol. C.) $\frac{b}{a} = \frac{d}{c}$. But $1 \pm \frac{b}{a} = 1 \pm \frac{d}{c}$, or $\frac{a \pm b}{a} = \frac{c \pm d}{c}$; therefore (Schol. C.) $a \pm b : a :: c \pm d : c$, or (3. Pr.) $a : a \pm b :: c : c \pm d$, which represents A : A ± B :: C : C ± D.

PROP. VIII.

IF the first has to the second the same ratio which the third has to the fourth, and the fifth to the second the same ratio which the sixth has to the fourth, the first and fifth together will have to the second the same ratio which the third and sixth together have to the fourth.

Let A : B :: C : D, and also E : B :: F : D: then A + E : B :: C + F : D.

Let P be a common measure of A, B, E, which are magnitudes of one sort; and take Q the same part of D, that P is of B; and because A, B, C, D are proportional, C will contain Q as often as A contains P; (10. Def. Pr.) in like manner, because E, B, F, D are proportional, F will contain Q as often as E contains P (10. Def. Pr.). And because A contains P as often as C contains Q; and that E contains P as often as F contains Q; therefore the sum A + E will contain P as often as the sum C + F contains Q. Now P and Q are like parts of B and D, and it has been shown that they are contained the same number of times in A + E and C + F; wherefore,



A + E : B :: C + F : D (10. Def. Pr.)

Algebraically. Let the proportions A : B :: C : D, and E : B :: F : D, be represented by $a : b :: c : d$ and $e : b :: f : d$, in numbers. Then, (Schol. C.) $\frac{a}{b} = \frac{c}{d}$, and

$\frac{e}{b} = \frac{f}{d}$; therefore, adding equals to equals, $\frac{a+e}{b} = \frac{c+f}{d}$, or $a + e : b :: c + f : d$, which represents

A + E : B :: C + F : D.

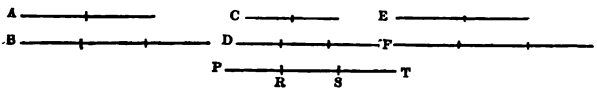
PROP. IX.

If any number of magnitudes be proportionals, as one of the antecedents is to its consequent, so are all the antecedents taken together, to all the consequents.

Let A : B :: C : D :: E : F, then

A : B :: A + C + E : B + D + F.

Let PR be a common measure of A and B, and take RS the same part of D, and ST the same part of F, that PR is of B: and because the ratios of A to B, and of C to D, and of E to



F, are all equal, C will contain RS, and E will contain ST, as often as A contains PR. (10. Def. Pr.) And because A and C and E contain their parts PR, RS, ST the same number of times, whatever number of times A contains PR, the same number of times will the sum A + C + E contain the sum PT. In like manner, because B and D and F contain their parts PR, RS, ST the same number of times, whatever number of times B contains PR, the same number of times will the sum B + D + F contain the sum PT. Wherefore PR and PT are like parts of B and B + D + F; and it has been shown that PR is contained in A as often as PT in A + C + E; consequently

A : B :: A + C + E : B + D + F. (10. Def. Pr.)

Algebraically. Let the proportion A : B :: C : D :: E : F be represented by $a : b :: c : d :: e : f$, in numbers. Then, (4. and 3. Pr.) $c : a :: d : b$ and $e : a :: f . b$; whence, (Schol. C.) $\frac{c}{a} = \frac{d}{b}$, and $\frac{e}{a} = \frac{f}{b}$. Adding equals to equals, $1 + \frac{c}{a} + \frac{e}{a} = 1 + \frac{d}{b} + \frac{f}{b}$, or $\frac{a+c+e}{a} = \frac{b+d+f}{b}$; therefore, (Schol. C.) $a + c + e : a :: b + d + f : b$, or (3. and 4. Pr.) $a : b :: a + c + e :$

$b + d + f$, which represents the proportion

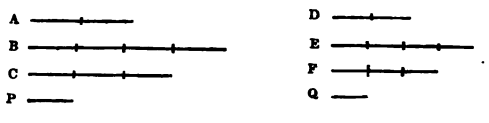
A . B :: A + C + E : B + D + F.

PROP. X.

If there be any number of magnitudes, and as many others, which, taken two and two in order, have the same ratio: the first of the one rank of magnitudes will have to the last, the same ratio which the first of the other rank has to the last.*

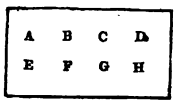
First, let there be three magnitudes A, B, C, and other three D, E, F, such that A : B :: D : E, and B : C :: E : F: then A : C :: D : F.

Let P be a common measure of A, B, and C, and take Q the same part of D that P is of A: and, because D, E, F are proportional to A, B, C, E and F will contain Q the same number of times respectively, that B and C contain P. (10. Def. Pr.) Consequently P and Q are like parts of C and F; and they are contained the same number of times in A and D; wherefore



A : C :: D : F. (10. Def. Pr.)

Again, let there be four magnitudes, and other four, which, taken two and two in order, have the same ratio, viz. A : B :: E : F, B : C :: F : G, C : D :: G : H, then A : D :: E : H.



For, the three A, B, C, and the three E, F, G, taken two and two in order, have the same ratio; wherefore, by the foregoing case, A : C :: E : G. And again, the three A, C, D, and the three E, G, H, taken two and two in order, have the same ratio; wherefore, as before, A : D :: E : H. In the same manner may the demonstration be extended to any number of magnitudes.

Cor. From this it follows that the ratios which are compounded of equal ratios, taken in the same order, are equal to one another. That is, for example, the ratio of A to D, which

* Two ranks of magnitudes are said to be proportional, taken two and two in order, when the ratio of the first to the second of one rank is equal to the ratio of the first to the second of the other rank; and the ratio of the second to the third of the first rank, equal to the ratio of the second to the third of the other rank, and so on in order.

is said to be compounded (16. Def. Pr.) of the ratios of A to B, B to C, &c., is equal to the ratio of E to H, which is said to be compounded of the ratios of E to F, F to G, &c.; and the several ratios of A to B, B to C, &c., are equal, in order, to the several ratios of E to F, F to G, &c.

Algebraically. Let the proportions A : B :: D : E, and B : C :: E : F, be represented in numbers by $a : b :: d : e$ and $b : c :: e : f$: then (Schol. C.) $\frac{a}{b} = \frac{d}{e}$, and $\frac{b}{c} = \frac{e}{f}$: multiplying equals by equals, $\frac{ba}{bc} = \frac{ed}{ef}$, or $\frac{a}{c} = \frac{d}{f}$; therefore (Schol. C.) $a : c :: d : f$, which represents the proportion A : C :: D : F.

Again, let the proportions A : B :: E : F, B : C :: F : G, and C : D :: G : H, be represented by $a : b :: e : f$, $b : c :: f : g$ and $c : d :: g : h$: then, (Schol. C.) $\frac{a}{b} = \frac{e}{f}$, $\frac{b}{c} = \frac{f}{g}$, and $\frac{c}{d} = \frac{g}{h}$: multiplying equals by equals, $\frac{abc}{bcd} = \frac{efg}{fgh}$, or $\frac{a}{d} = \frac{e}{h}$; therefore $a : d :: e : h$, which represents the proportion A : D :: E : H.

N.B. When a proportion is inferred, according to this Theorem, it is said to be inferred *by equality of distance, directly, or in ordinate proportion.* The proposition is usually cited by the words, *ex æquali* (sc. distantia), or *ex æquo.*

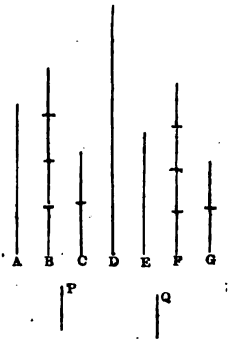
PROP. XI.

If there be any number of magnitudes, and as many others, which, taken two and two in a cross order, have the same ratio: the first of the one set of magnitudes will have to the last, the same ratio which the first of the other set has to the last.*

First, let there be three magnitudes A, B, C, and other

* Two ranks of quantities are said to be proportional, taken two and two in a cross order, when the ratio of the first to the second of one rank, is equal to the ratio of the last but one to the last of the other rank; and the ratio of the second to the third of the first rank, equal to the ratio of the last but two to the last but one of the other rank; and the ratio of the third to the fourth of the first rank equal to the ratio of the last but three to the last but two of the other rank, and so on in a cross order.

three D, E, F, which, taken two and two in a cross order have the same ratio, viz. A : B :: E : F, and B : C :: D : E: then A : C :: D : F.



Let P measure both B and C, and let Q be the same part of F that P is of B; also take G to contain Q as often as C contains P. It is manifest that

B : C :: F : G (10. Def. Pr.)

but, by hypothesis,

B : C :: D : E.

Wherefore, by equality of ratios,

D : E :: F : G;

and, alternately, D : F :: E : G (4. Pr.).

Now A : B :: E : F, and B : C :: F : G, therefore, ex æquali, in ordinate proportion,

A : C :: E : G (10. Pr.);

but it has been proved that

D : F :: E : G,

wherefore, by equality of ratios,

A : C :: D : F.

Next, let there be four magnitudes A, B, C, D, and other four, E, F, G, H, which, taken two and two in a cross order, have the same ratio, viz. A : B :: G : H, B : C :: F : G, and C : D :: E : F: then, A : D :: E : H.

A	B	C	D
E	F	G	H

For, the three A, B, C, and the three F, G, H, taken two and two in a cross order, have the same ratio; wherefore, by the foregoing case, A : C :: F : H. And again, the three A, C, D, and the three E, F, H, taken two and two in a cross order, have the same ratio; wherefore, as before, A : D :: E : H. In the same manner may the demonstration be extended to any number of magnitudes.

Algebraically. Let the proportions A : B :: E : F, and B : C :: D : E, be represented in numbers by $a : b :: e : f$, and $b : c :: d : e$: then, (Schol. C.) $\frac{a}{b} = \frac{e}{f}$, and $\frac{b}{c} = \frac{d}{e}$: multiplying equals by equals, $\frac{ab}{bc} = \frac{ed}{ef}$, or $\frac{a}{c} = \frac{d}{f}$; therefore, (Schol. C.) $a : c :: d : f$, which represents the proportion A : C :: D : F.

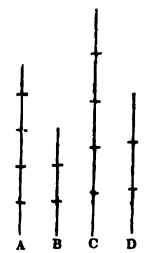
In like manner the proposition may be proved when there are four magnitudes and four others, which, taken two and two in a cross order, are proportional.

N.B. This proposition is usually cited by the words, *ex æquali in proportione perturbata*, or ex æquo perturbate.

PROP. XII.

If four magnitudes be proportional, as the sum of the two first is to their difference, so is the sum of the other two to their difference.

Let A : B :: C : D, and let A be greater than B, and consequently C greater than D: then A + B : A − B :: C + D : C − D. For, by division, A − B : B :: C − D : D (6. Pr.) and by inversion, B : A − B :: D : C − D (3. Pr.). But by composition, A + B : B :: C + D : D (5. Pr.), therefore, ex æquali, A + B : A − B :: C + D : C − D (10. Pr.)



And if B be greater than A, and consequently D greater than C, it is proved in the same manner, that A + B : B − A :: C + D : D − C.

Algebraically. Let the proportion A : B :: C : D be represented in numbers by $a : b :: c : d$; then (5. Pr.) $\frac{a+b}{b} = \frac{c+d}{d}$, and (6. Pr.) $\frac{a-b}{b} = \frac{c-d}{d}$: dividing equals by equals, $\frac{a+b}{a-b} = \frac{c+d}{c-d}$; therefore (Schol. C.) $a + b : a - b :: c + d : c - d$, which represents the proportion A + B : A − B :: C + D : C − D.

N.B. A proportion inferred in this manner, is said to be inferred *by mixing*.

PROP. XIII.

A ratio which is compounded of other ratios that are expressed by numbers, is equal to the ratio of the product of all the antecedents to the product of all the consequents.

First, let there be three magnitudes, H, K, L, and let H : K :: a : b, and K : L :: c : d; a, b, c, d denoting numbers: then the ratio of H to L, which is said to be compounded of the ratios of H to K, and of K to L (16. Def. Pr.), is equal to the ratio of ac to bd.

For, because numbers have the same ratio as their equimultiples (1. Pr.), therefore

$$a : b :: ac : bc,$$
$$\text{and } e : d :: bc : bd.$$

It is therefore evident that the three magnitudes H, K, L, and the three products ac, bc, and bd, have the same ratio, if taken two and two in order; consequently, ex æquali,

$$\text{H} : \text{L} :: ac : bd \text{ (10. Pr.)}$$

Next, let there be four magnitudes, H, K, L, M, and let $\text{H} : \text{K} :: a : b$, $\text{K} : \text{L} :: c : d$, $\text{L} : \text{M} :: e : f$; a, b, c, d, e, f denoting numbers: then the ratio of H to M, which is said to be compounded of the ratios of H to K, and of K to L, and of L to M, is equal to the ratio of ace to bdf.

For, $a : b :: ace : bce$. (1. Pr.).
$c : d :: bce : bde$.
$e : f :: bde : bdf$.

Wherefore the four magnitudes H, K, L, M, and the four products ace, bce, bde, bdf, taken two and two in order, have the same ratio; consequently, ex æquali,

$$\text{H} : \text{M} :: ace : bdf \text{ (10. Pr.)}$$

The same mode of reasoning will apply to any number of magnitudes, and to any number of component ratios.

Cor. Hence it is easy to infer, that two ratios, each of which is compounded of the same number of ratios that are equal each to each, taken in any order whatever, are equivalent to one another. For the component ratios will, in both cases, be equal to the same numerical ratios, each to each, although not in the same order; and, by this proposition, each of the compound ratios will be equivalent to the ratio of the product of all the antecedents of the numerical ratios to the product of all the consequents.

PROP. XIV.

The duplicate ratio of two numbers is equal to the ratio of their squares.

Let a and b be two numbers, and let $a : b :: b : c$: then the ratio of a to c, which is said to be duplicate of the ratio of a to b (17. Def. Pr.), is equal to the ratio of a^2 to b^2.

For, $a : b :: a^2 : ab$, (1. Pr.)
and $b : c (:: a : b) :: ab : b^2$.

Wherefore the three numbers, a, b, c, and the other three, a^2, ab, b^2, taken two and two in order, have the same ratio: consequently, ex æquali,

$$a : c :: a^2 : b^2. \text{ (10. Pr.)}$$

PROP. XV.

THE triplicate ratio of two numbers is equal to the ratio of their cubes.

Let a and b be two numbers, and let $a : b :: b : c :: c : d$, then the ratio of a to d, which is said to be triplicate of the ratio a to b (18. Def. Pr.), is equal to the ratio of a^3 to b^3.

For, $a : b :: a^3 : a^2 b$, (1. Pr.)
$b : c (:: a : b) :: a^2 b : ab^2$
$c : d (:: a : b) :: ab^2 : b^3$.

Wherefore the four numbers, a, b, c, d, and the other four, a^3, a^2b, ab^2, b^3, taken two and two in order, have the same ratio: consequently, ex æquali, $a : d :: a^3 : b^3$. (10. Pr.)

PROP. XVI.

IF there be any proportions in numbers: the products of their like terms will be proportional.

Let $a : b :: c : d$,
and $e : f :: g : h$;
then $ae : bf :: cg : dh$.
For because $a : b :: c : d$,
therefore $ae : be :: cg : dg$. (1. Pr.)
And because $e : f :: g : h$,
therefore $be : bf :: dg : dh$;

consequently, ex æquali, $ae : bf :: cg : dh$.

The same demonstration will apply to any number of proportions.

PROP. XVII.

IF four numbers be proportional, the product of the extremes is equal to the product of the means; and if the product of the extremes be equal to the product of the means, the four numbers are proportional.

Let $a : b :: c : d$; then $ad = bc$.
For, $a : b :: ad : bd$, (1. Pr.)
and $c : d :: bc : bd$:

but, by hypothesis, $a : b :: c : d$,

therefore, by equality of ratios,

$$ad : bd :: bc : bd;$$

therefore $ad = bc$. (4 Cor. Def. Pr.)

Again, if $ad = bc$; then $a : b :: c : d$.

For, because $ad = bc$,

$$ad : bd :: bc : bd;$$ (3 Cor. Def. Pr.)

but $ad : bd :: a : b$, (Cor. 1. Pr.)

and $bc : bd :: c : d$;

therefore, by equality of ratios

$$a : b :: c : d.$$

Cor. If three numbers be continual proportionals, the product of the extremes is equal to the square of the middle term; and if the product of the extremes is equal to the square of the middle term, the three numbers are proportionals.

THIRD GENERAL THEOREM.

Let there be two geometrical magnitudes A and B, of the same kind, which cannot be directly compared together; also let $A_,$ $B_,$ A', B', be other geometrical magnitudes of the same kind, such that in the two series

$$A_, \quad A \quad A'$$
$$B_, \quad B \quad B',$$

the excesses of A above $A_,$ and of A' above A, also the excesses of B above $B_,$ and of B' above B may be severally less than any magnitude that can be assigned, however small; the ratios of $A_,$ to $B_,$ and of A' to B' being equal, suppose the same as that of m to n (where m and n are two magnitudes of a like kind): it is required to prove that the ratio of A to B is also equal to that of m to n.

If A be not to B as m to n, let m be to n as A to S, in which proportion S is either greater or less than B: first let S be less than B, and let the difference between them be some assignable quantity, great or small; then S is less than $B_,$ or $B_,$ is greater than S.

Now $m : n :: A_, : B_,$ and $m : n :: A : S$;

hence, $A_, : B_, :: A : S$, or alternately, $A_, : A :: B_, : S$.

But by hypothesis, $A_,$ is less than A; therefore $B_,$ is less than S (2. Cor. Def. Pr.), which is absurd. Therefore m is not to n as A is to a magnitude less than B by any assignable magnitude, great or small.

Next, let S be greater than B, and let the difference between them be some assignable magnitude, great or small; then S is greater than B', or B' is less than S.

Now $m : n :: A' : B'$, and $m : n :: A : S$;

hence $A' : B' :: A : S$, or alternately, $A' : A :: B' : S$.

But by hypothesis, A' is greater than A; therefore B' is greater than S, which is absurd: therefore m is not to n as A is to a magnitude greater than B, by any assignable magnitude, great or small.

Moreover, from the hypothesis, m is not to n as A to B, nor as A to B': and both B, and B' differ from B by magnitudes less than any that can be assigned; therefore it may be concluded that m is to n as A to B. Q. E. D.

ELEMENTS

OF

GEOMETRY.

EUCLID. BOOK VI.

DEFINITIONS.

I.

SIMILAR rectilineal figures are those which have their several angles equal, each to each, and the sides about the equal angles proportionals.

II.

"Reciprocal figures, viz. triangles and parallelograms, are "such as have their sides about two of their angles pro- "portionals, in such manner that a side of the first figure "is to a side of the other, as the remaining side of this other "is to the remaining side of the first."

III.

A straight line is said to be cut in extreme and mean ratio, when the whole is to the greater segment, as the greater segment is to the less.

IV.

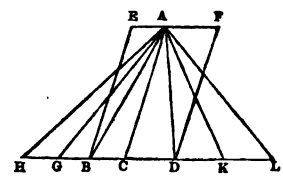
The altitude of any figure is the straight line drawn from its vertex perpendicular to the base.

PROP. I. THEOR.

TRIANGLES and parallelograms of the same altitude are one to another as their bases.

Let the triangles ABC, ACD, and the parallelograms EC, CF have the same altitude, viz. the perpendicular drawn from the point A to BD: then, as the base BC, is to the base CD, so is the triangle ABC to the triangle ACD, and the parallelogram EC to the parallelogram CF.

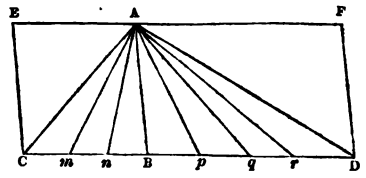
Produce BD both ways to the points H, L, and take any number of straight lines BG, GH, each equal to the base BC; and DK, KL, any number of them, each equal to the base CD; and draw AG, AH, AK, AL. Then, because CB, BG, GH are all equal, the triangles AHG, AGB, ABC are all equivalent (38. 1.): therefore, whatever multiple the base HC is of the base BC, the same multiple is the triangle AHC of the triangle ABC: for the same reason, whatever multiple the base LC is of the base CD, the same multiple is the triangle ALC of the triangle ADC: and if the base HC be equal to the base CL, the triangle AHC is also equivalent to the triangle ALC (38. 1.); and if the base HC be greater than the base CL, likewise the triangle AHC is greater than the triangle ALC; and if less, less: therefore, since there are four magnitudes, viz. the two bases BC, CD, and the two triangles, ABC, ACD; and of the base BC and the triangle ABC, the first and third, any equimultiples whatever have been taken, viz. the base HC and triangle AHC, and of the base CD and triangle ACD, the second and fourth, have been taken any equimultiples whatever, viz. the base CL and triangle ALC: and that it has been shown that, if the base HC be greater than the base CL, the triangle AHC is greater than the triangle ALC; and if equal, equal; and if less, less; therefore (5. Def. 5. Euclid, or 10. Def. Pr.), as the base BC is to the base CD, so is the triangle ABC to the triangle ACD.



And because the parallelogram CE is double of the triangle ABC (41. 1.), and the parallelogram CF double of the triangle ACD, and that magnitudes have the same ratio which their equimultiples have (1. Pr.); as the triangle ABC is to the triangle ACD, so is the parallelogram EC to the parallelogram CF: and because it has been shown that, as the base BC is to the base CD, so is the triangle ABC to the triangle ACD; and as the triangle ABC is to the triangle ACD, so is the parallelogram EC to the parallelogram CF: therefore, as the base BC is to the base CD, so is (5. Cor. Def. Pr.) the parallelogram EC to the parallelogram CF. Wherefore, triangles, &c. Q. E. D.

Otherwise,

Let the triangles ABC, ABD, and the parallelograms BE and BF, have the same altitude, viz. the perpendicular drawn from A upon the base CD: then, as the base BC to the base BD, so is the triangle ABC to the triangle ABD, and the parallelogram BE to the parallelogram BF.



If the bases BC and BD have a common measure, each of them may be divided into parts equal to that common measure. Let C*m*, *mn*, *n*B be the parts into which CB is divided; and B*p*, *pq*, *qr*, *r*D, the parts into which BD is divided; and draw A*m*, A*n*, A*p*, A*q*, A*r*. Then the triangles AC*m*, A*mn*, &c. are all equivalent to one another, because they have equal bases and are between the same parallels (38. 1.). Wherefore the base BC contains the line C*m* as often as the triangle ACB contains the triangle AC*m*; and the base BD contains the line C*m* as often as the triangle ABD contains the triangle AC*m*. Consequently, the line C*m* and the triangle AC*m* are like parts of the base BD and of the triangle ABD; and it has been shown that those parts are contained the same number of times in the base BC and in the triangle ABC; wherefore,

ABC : ABD :: CB : BD. (10. Def. Pr.)

And because the parallelogram BE is double of the triangle ABC (41. 1.), and the parallelogram BF double of the triangle ABD; and because quantities have the same ratio which their equimultiples have (1. Pr.); therefore,

ABC : ABD :: BE : BF.

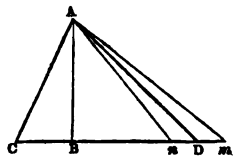
But it has been proved, that

ABC : ABD :: BC : BD,

therefore, by equality of ratios,

BE : BF :: BC : BD.

If CB and BD, the bases of the triangles, are incommensurable, by continually bisecting one of them, as CB, there may at length (D. Pr.) be found a line equal to or less than the least that can be assigned: this part of CB will be contained in BD a certain number of times with a remainder as D*n*, or an excess, as D*m*, less than itself; that is, less than any line that can be assigned. Join A, *n* and A, *m*: then, the bases CB and B*n*, CB and B*m* being commensurable,



triangle ABC : triangle ABn :: CB : Bn,
and triangle ABC : triangle ABm :: CB : Bm.

But the difference between Bn and Bm is equal to, or less than, the least line that can be assigned; and the difference between the triangles ABn and ABm, being equivalent to a rectangle whose base is the half of mn, and whose altitude is equal to that of the triangle ACB, or ABD, is (4. Cor. A. Pr.) less than the least surface that can be assigned: therefore, by the Second General Theorem, (Proportion,)

triangle ABC : triangle ABD :: CB : BD.

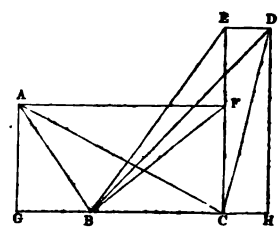
Cor. 1. Triangles and parallelograms that have equal altitudes are to one another as their bases.

For, if the figures be placed so as to have their bases in one straight line, they will be between the same parallels (33. 1.) because the perpendiculars drawn from the vertices of the triangles to the bases are both equal and parallel. Then the demonstration will be the same as before.

Cor. 2. Triangles which have the same bases are to one another as their altitudes.

Let ABC, DBC be two triangles having the same base BC, they shall be to one another as the altitudes AG and DH, which (4. Def. 6.) are perpendiculars let fall from A and D upon BC produced if necessary either way.

Through C draw CE perpendicular to BC, and through A and D draw AF, DE parallel to BC; also join B, F and B, E. Then the triangles ABC, DBC will be respectively equivalent to FBC and EBC, because they stand on the same base BC and are between the same parallels respectively. Since AG, DH, EC are perpendicular to BC, they are parallel to one another (28. 1): also by construction AF and DE are parallel to BC; therefore (34. 1.) FG and EH are parallelograms. Consequently AG is equal to FC, and DH to EC; therefore also CF and CE are respectively equal to the altitudes of the triangles ABC and DBC. Now, considering FBC and EBC as two triangles having bases CF and CE in the same straight line, and having a common altitude BC,



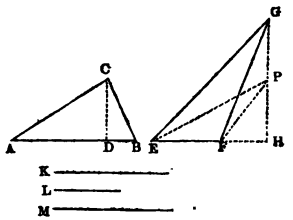
FBC : EBC :: CF : CE by the proposition;
that is, ABC : DBC :: CF : CE.

And in a similar manner it may be proved that triangles upon equal bases, and parallelograms upon the same, or equal bases are to one another as their altitudes.

Cor. 3. Triangles and parallelograms are to one another

in a ratio compounded of the ratios of their bases and altitudes.

Let ABC, EFG be two triangles whose bases are AB and EF, and whose altitudes (4. Def. 6.) are CD and GH: it is required to prove that the triangles are to one another in a ratio compounded of AB to EF, and of CD to GH.



In GH, the greater of the two perpendiculars, take HP equal to CD, the less, and draw EP, FP. Let K be any line, and make AB to EF as K to L; also make HP, or CD to GH as L to M. Then the ratio of K to M is that which is compounded of the ratios of AB to EF, and of CD to GH; that is, of the bases and altitudes of the triangles.

Now ACB : EPF :: AB : EF (1. 6.) or :: K : L,

and EPF : GEF :: CD : GH (2. Cor. 1. 6.) or :: L : M.

Therefore, ex æquali, ACB : GEF :: K : M.

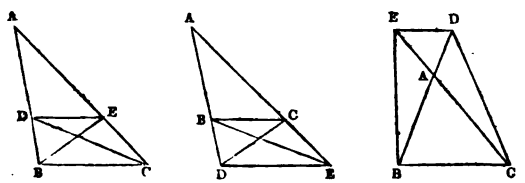
That is, the triangles ACB, GEF are to one another in a ratio compounded of the ratios of the bases and altitudes.

Since parallelograms having bases and altitudes equal to those of the triangles have the same ratio to one another that the triangles have (being the doubles of them) they, also, are to one another in a ratio compounded of the ratios of their bases and altitudes.

PROP. II. THEOR.

If a straight line be drawn parallel to one of the sides of a triangle, it shall cut the other sides, or these produced, proportionally: and if the sides, or the sides produced, be cut proportionally, the straight line which joins the points of section shall be parallel to the remaining side of the triangle.

Let DE be drawn parallel to BC, one of the sides of the triangle ABC: BD is to DA, as CE to EA.



Join B, E and C, D; then the triangle BDE is equivalent to the triangle CDE, (37. 1.) because they are upon the same base DE, and between the same parallels DE, BC: ADE is another triangle, and equivalent magnitudes have to the same, the same ratio; (3. Cor. Def. Pr.) therefore, as the triangle BDE to the triangle ADE, so is the triangle CDE to the triangle ADE; but as the triangle BDE to the triangle ADE, so is (1. 6.) BD to DA, because having the same altitude, viz. the perpendicular drawn from the point E to AB, they are to one another as their bases; and for the same reason, as the triangle CDE to the triangle ADE, so is CE to EA: therefore, as BD to DA, so is CE to EA (5. Cor. Def. Pr.).

Next, let the sides AB, AC of the triangle ABC, or these produced, be cut proportionally in the points D, E, that is, so that BD be to DA, as CE to EA, and join D, E; DE is parallel to BC.

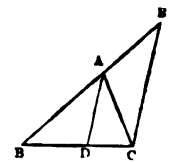
The same construction being made, because as BD to DA, so is CE to EA; and as BD to DA, so is the triangle BDE to the triangle ADE (1. 6.); and as CE to EA, so is the triangle CDE to the triangle ADE: therefore the triangle BDE is to the triangle ADE, as the triangle CDE to the triangle ADE; that is, the triangles BDE, CDE have the same ratio to the triangle ADE; and therefore (4. Cor. Def. Pr.) the triangle BDE is equivalent to the triangle CDE; and they are on the same base DE; but equivalent triangles on the same base are between the same parallels (39. 1.); therefore DE is parallel to BC. Wherefore, if a straight line, &c. Q. E. D.

PROP. III. THEOR.

IF the angle of a triangle be divided into two equal angles, by a straight line which also cuts the base, the segments of the base shall have the same ratio which the other sides of the triangle have to one another; and if the segments of the base have the same ratio which the other sides of the triangle have to one another, the straight line drawn from the vertex to the point of section divides the vertical angle into two equal angles.

Let the angle BAC of any triangle ABC be divided into two equal angles by the straight line AD: BD is to DC, as BA to AC.

Through the point C draw CE parallel (31. 1.) to DA, and



let BA produced meet CE in E. Because the straight line AC meets the parallels AD, EC the angle ACE is equal to the alternate angle CAD (29. 1.): but CAD, by the hypothesis, is equal to the angle BAD; wherefore BAD is equal to the angle ACE. Again, because the straight line BAE meets the parallels AD, EC, the outward angle BAD is equal to the inward and opposite angle AEC; but the angle ACE has been proved equal to the angle BAD; therefore also ACE is equal to the angle AEC, and consequently the side AE is equal to the side (6. 1.) AC: and because AD is drawn parallel to one of the sides of the triangle BCE, viz. to EC, BD is to DC, as BA to AE, (2. 6.) but AE is equal to AC; therefore, as BD to DC, so is BA to AC (3. Cor. Def. Pr.).

Let now BD be to DC, as BA to AC, and join A, D; the angle BAC is divided into two equal angles by the straight line AD.

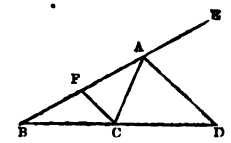
The same construction being made; because (Hyp.), BD is to DC, as BA to AC; and that BD is to DC, as BA to AE, (2. 6.) since AD is parallel to EC; therefore BA is to AC, as BA to AE. (5. Cor. Def. Pr.) Consequently AC is equal to AE (4 Cor. Def. Pr.), and the angle AEC is therefore equal to the angle ACE (5. 1.): but the angle AEC is equal to the outward and opposite angle BAD; and the angle ACE is equal to the alternate angle CAD (29. 1.): wherefore also the angle BAD is equal to the angle CAD: therefore the angle BAC is cut into two equal angles by the straight line AD. Therefore, if the angle, &c. Q. E. D.

PROP. A. THEOR.

If the outward angle of a triangle, made by producing one of its sides, be divided into two equal angles, by a straight line which also cuts the base produced; the segments between the dividing line and the extremities of the base have the same ratio which the other sides of the triangle have to one another: and if the segments of the base produced have the same ratio which the other sides of the triangle have; the straight line drawn from the vertex to the point of section divides the outward angle of the triangle into two equal angles.

Let the outward angle CAE of any triangle ABC be divided into two equal angles by the straight line AD which meets the base produced in D: BD is to DC, as BA to AC.

Through C draw CF parallel to AD (31. 1.): and because the straight line AC meets the parallels AD, FC, the angle ACF is equal to the alternate angle CAD (29. 1.); but CAD is equal to the angle DAE (Hyp.); therefore also DAE is equal to the angle ACF. Again, because the straight line FAE meets the parallels AD, FC, the outward angle DAE is equal to the inward and opposite angle CFA; but the angle ACF has been proved equal to the angle DAE; therefore also the angle ACF is equal to the angle CFA, and consequently the side AF is equal to the side AC (6. 1.): and because AD is parallel to FC, a side of the triangle BCF, BD is to DC, as BA to AF (2. 6.), but AF is equal to AC; as therefore BD is to DC, so is BA to AC.



Let now BD be to DC, as BA to AC, and join A, D; the angle CAD is equal to the angle DAE.

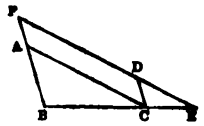
The same construction being made; because BD is to DC, as BA to AC, and that BD is also to DC, as BA to AF; therefore (5 Cor. Def. Pr.) BA is to AC, as BA to AF: wherefore AC is equal to AF (4. Cor. Def. Pr.), and the angle AFC equal (5. 1.) to the angle ACF: but the angle AFC is equal to the outward angle EAD, and the angle ACF to the alternate angle CAD; therefore also EAD is equal to the angle CAD. Wherefore, if the outward, &c. Q. E. D.

PROP. IV. THEOR.

THE sides about the equal angles of equiangular triangles are proportionals: and those which are opposite to the equal angles are homologous sides, that is, the antecedents or consequents of the ratios.

Let ABC, DCE be equiangular triangles, having the angle ABC equal to the angle DCE, and the angle ACB to the angle DEC; and consequently (32. 1.) the angle BAC equal to the angle CDE: the sides about the equal angles of the triangles ABC, CDE are proportionals, and those are the homologous sides which are opposite to the equal angles.

Let the triangle DCE be placed, so that its side CE may be contiguous to BC, and in the same straight line with it.



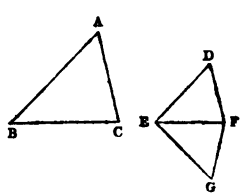
And because the angles ABC, ACB are together less than two right angles (17. 1.), ABC, and DEC, which is equal to ACB, are also less than two right angles; wherefore BA, ED produced shall meet (5. Post.); let them be produced and meet in the point F; and because the angle ABC is equal to the angle DCE, BF is parallel (28. 1.) to CD. Again, because the angle ACB is equal to the angle DEC, AC is parallel to FE (28. 1.): therefore FACD is a parallelogram, and consequently AF is equal to CD, and AC to FD (34. 1.): and because AC is parallel to FE, one of the sides of the triangle FBE, BC is to CE, as BA to AF (2. 6.): but AF is equal to CD; therefore (3. Cor. Def. Pr.) as BC is to CE, so is BA to CD. Again, because CD is parallel to BF, as BC is to CE, so is FD to DE (2. 6.); but FD is equal to AC; therefore, as BC is to CE, so is AC to DE. Therefore, by equality of ratios (5. Cor. Def. Pr.), as BA is to CD, so is AC to DE; and alternately, as BA to AC, so is CD to DE. Therefore, the sides, &c. Q. E. D.

PROP. V. THEOR.

If the sides of two triangles, about each of their angles, be proportionals: the triangles shall be equiangular, and have their equal angles opposite to the homologous sides.

Let the triangles ABC, DEF have their sides proportionals, so that AB is to BC, as DE to EF; and BC to CA, as EF to FD; and consequently, ex æquali, (10. Pr.) BA to AC, as ED to DF: the triangle ABC is equiangular to the triangle DEF, and their equal angles are opposite to the homologous sides, viz. the angle ABC equal to the angle DEF, and BCA to EFD, and also BAC to EDF.

At the points E, F, in the straight line EF, make (23. 1.) the angle FEG equal to the angle ABC, and the angle EFG equal to BCA: wherefore the remaining angle BAC is equal to the remaining angle EGF (32. 1.), and the triangle ABC is therefore equiangular to the triangle GEF; and consequently they have their sides opposite to the equal angles proportionals (4. 6.). Wherefore, as AB to BC, so is GE to



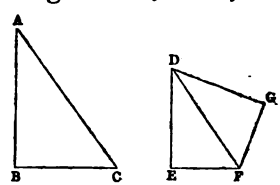
EF: but as AB to BC, so is DE to EF; therefore as DE to EF, so (5. Cor. Def. Pr.) GE to EF: therefore DE and GE have the same ratio to EF, and consequently are equal (4. Cor. Def. Pr.): for the same reason DF is equal to FG. And because, in the triangles DEF, GEF, DE is equal to EG, and EF common, the two sides DE, EF are equal to the two GE, EF, and the base DF is equal to the base GF; therefore the angle DEF is equal (8. 1.) to the angle GEF, and the other angles to the other angles which are subtended by the equal sides (4. 1.): wherefore the angle DFE is equal to the angle GFE, and EDF to EGF: and because the angle DEF is equal to the angle GEF, and GEF to the angle ABC; therefore the angle ABC is equal to the angle DEF: for the same reason, the angle ACB is equal to the angle DFE, and the angle at A to the angle at D: therefore, the triangle ABC is equiangular to the triangle DEF. Wherefore, if the sides, &c. Q. E. D.

PROP. VI. THEOR.

If two triangles have one angle of the one equal to one angle of the other, and the sides about the equal angles proportionals: the triangles shall be equiangular, and shall have those angles equal which are opposite to the homologous sides.

Let the triangles ABC, DEF have the angle BAC in the one equal to the angle EDF in the other, and the sides about those angles proportionals; that is, BA to AC, as ED to DF: the triangles ABC, DEF are equiangular, and have the angle ABC equal to the angle DEF, and ACB to DFE.

At the points D, F, in the straight line DF, make (23. 1.) the angle FDG equal to either of the angles BAC, EDF; and the angle DFG equal to the angle ACB: wherefore the remaining angle at B is equal to the remaining one at G (32. 1.), and consequently the triangle ABC is equiangular to the triangle DGF; and therefore as BA to AC, so is (4. 6.) GD to DF; but by the hypothesis, as BA to AC, so is ED to DF; as therefore ED to DF, so is (5. Cor. Def. Pr.) GD to DF; wherefore ED is equal (4. Cor. Def. Pr.) to DG; and DF is common to the two triangles EDF, GDF: therefore the two sides ED, DF are equal to the two sides GD, DF; and the



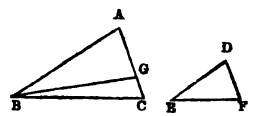
angle EDF is equal to the angle GDF; wherefore the base EF is equal to the base FG (4. 1.), and the triangle EDF to the triangle GDF, and the remaining angles to the remaining angles, each to each, which are subtended by the equal sides; therefore the angle DFG is equal to the angle DFE, and the angle at G to the angle at E: but the angle DFG is equal to the angle ACB; therefore the angle ACB is equal to the angle DFE: and the angle BAC is equal to the angle EDF (Hyp.); wherefore also the remaining angle at B is equal to the remaining angle at E: therefore, the triangle ABC is equiangular to the triangle DEF. Wherefore, if two triangles, &c. Q.E.D.

PROP. VII. THEOR.

IF two triangles have one angle of the one equal to one angle of the other, and the sides about two other angles proportionals, then, if each of the remaining angles be either less or not less than a right angle, or if one of them be a right angle, the triangles shall be equiangular, and have those angles equal about which the sides are proportionals.

Let the two triangles ABC, DEF have one angle in the one equal to one angle in the other, viz. the angle BAC to the angle EDF, and the sides about two other angles ABC, DEF proportionals, so that AB is to BC, as DE to EF; and, in the first case, let each of the remaining angles at C, F be less than a right angle: the triangle ABC is equiangular to the triangle DEF, viz. the angle ABC is equal to the angle DEF, and the remaining angle at C to the remaining angle at F.

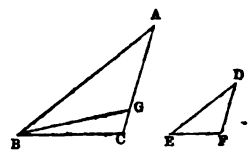
For if the angles ABC, DEF be not equal, one of them is greater than the other: let ABC be the greater; and at the point B, in the straight line AB, make the angle ABG equal to the angle (23. 1.) DEF. And because the angle at A is equal to the angle at D, and the angle ABG to the angle DEF; the remaining angle AGB is equal (32. 1.) to the remaining angle DFE: therefore the triangle ABG is equiangular to the triangle DEF; wherefore (4.6.) as AB is to BG, so is DE to EF: but as DE to EF, so by hypothesis, is AB to BC; therefore as AB to BC, so is AB to BG (5 Cor. Def. Pr.). And because AB has the same ratio to each of the lines BC, BG;



BC is equal (4. Cor. Def. Pr.) to BG, and therefore the angle BGC is equal to the angle BCG (5. 1.): but the angle BCG is, by hypothesis, less than a right angle; therefore also the angle BGC is less than a right angle, and the adjacent angle AGB must be greater than a right angle (13. 1.). But it was proved that the angle AGB is equal to the angle at F; therefore the angle at F is greater than a right angle: but, by the hypothesis, it is less than a right angle; which is absurd. Therefore the angles ABC, DEF are not unequal, that is, they are equal: and the angle at A is equal to the angle at D; wherefore the remaining angle at C is equal to the remaining angle at F; therefore the triangle ABC is equiangular to the triangle DEF.

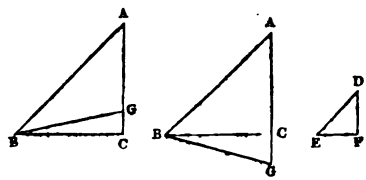
Next, let each of the angles at C, F be not less than a right angle: the triangle ABC is also in this case equiangular to the triangle DEF.

The same construction being made, it may be proved in like manner that BC is equal to BG, and the angle at C equal to the angle BGC: but the angle at C is not less than a right angle; therefore the angle BGC is not less than a right angle: wherefore two angles of the triangle BGC are together not less than two right angles, which is impossible (17. 1.); and therefore the triangle ABC may be proved to be equiangular to the triangle DEF, as in the first case.



Lastly, let one of the angles at C, F, viz. the angle at C, be a right angle: in this case likewise the triangle ABC is equiangular to the triangle DEF.

For if they be not equiangular, make, at the point B of the straight line AB, the angle ABG equal to the angle DEF; then it may be proved, as in the first case, that BG is equal to BC: but the angle BCG is a right angle, therefore (5. 1.) the angle BGC is also a right angle; whence two of the angles of the triangle BGC are together not less than two right angles, which is impossible (17. 1.): therefore the triangle ABC is equiangular to the triangle DEF. Wherefore, if the two triangles, &c. Q. E. D.

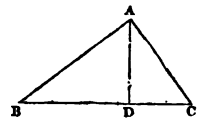


PROP. VIII. THEOR.

In a right-angled triangle, if a perpendicular be drawn from the right angle to the base, the triangles on each side of it are similar to the whole triangle, and to one another.

Let ABC be a right-angled triangle, having the right angle BAC; and from the point A let AD be drawn perpendicular to the base BC: the triangles ABD, ADC are similar to the whole triangle ABC, and to one another.

Because the angle BAC is equal to the angle ADB, each of them being a right angle, and that the angle at B is common to the two triangles ABC, ABD; the remaining angle ACB is equal to the remaining angle BAD (32. 1.): therefore the triangle ABC is equiangular to the triangle ABD, and the sides about their equal angles are proportionals (4. 6.); wherefore the triangles are similar (1. Def. 6.). In the like manner it may be demonstrated that the triangle ADC is equiangular and similar to the triangle ABC. But the triangles ABD, ACD being equiangular to ABC, and the sides about the equal angles in all being proportionals; it follows that those triangles are equiangular and similar to each other. Therefore, in a right-angled, &c. Q. E. D.



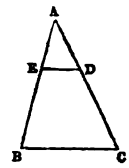
Cor. From this it is manifest that the perpendicular drawn from the right angle of a right-angled triangle to the base is a mean proportional between the segments of the base: and also, that each of the sides is a mean proportional between the base, and its segment adjacent to that side: because in the triangles BDA, ADC, BD is to DA, as DA to DC (4. 6.) and in the triangles ABC, DBA, BC is to BA, as BA to BD; and in the triangles ABC, ACD, BC is to CA, as CA to CD.

PROP. IX. PROB.

From a given straight line to cut off any part required.

Let AB be the given straight line: it is required to cut off any part from it.

From the point A draw a straight line AC, making any angle with AB; and in AC take any point D, and take AC the same multiple of AD, that AB is of the part which is to be cut off from it; join B, C, and draw DE parallel to it (31. 1.): then AE is the part required to be cut off.

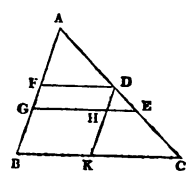


Because ED is parallel to one of the sides of the triangle ABC, viz. to BC, as CD is to DA, so is (2. 6.) BE to EA; and by composition (5. Pr.) CA is to AD, as BA to AE. But by construction, CA is a multiple of AD; that is, AD is a certain part of CA; therefore (Cor. 1. Pr.), AE is the like part of AB. Wherefore, from the straight line AB the part required is cut off. Q. E. F.

PROP. X. PROB.

To divide a given straight line similarly to a given divided straight line, that is, into parts that shall have the same ratios to one another which the parts of the divided given straight line have.

Let AB be the straight line given to be divided, and AC the divided line: it is required to divide AB similarly to AC. Let AC be divided in the points D, E: and let AB, AC be placed so as to contain any angle, and join B, C, and through the points D, E draw (31. 1.) DF, EG parallels to it; and through D draw DHK parallel to AB: therefore each of the figures FH, HB is a parallelogram: wherefore DH is equal (34. 1.) to FG, and HK to GB. And because HE is parallel to KC, one of the sides of the triangle DKC, as CE to ED, so is (2. 6.) KH to HD: but KH is equal to BG, and HD to GF: therefore, as CE to ED, so is BG to GF. Again, because FD is parallel to EG, one of the sides of the triangle AGE, as ED to DA, so is GF to FA: but it has been proved that CE is to ED, as BG to GF; and as ED to DA, so GF to FA: therefore the given straight line AB is divided similarly to AC. Q. E. F.

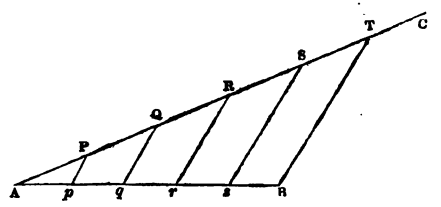


N.B. By a process founded on this proposition, a given straight line may be divided into any number of equal parts.

Let AB be the given line, and let it be required to divide it into any number of equal parts.

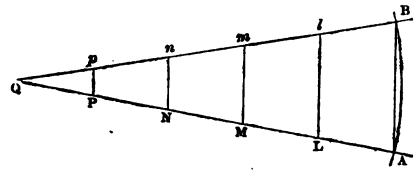
From A draw a line, as AC making with AB any angle whatever, and

on AC set equal spaces AP, PQ, &c., of any convenient length, as many in number as the parts into which AB is to be divided: join B and the last point T, of the divisions on AC, and draw P*p*, Q*q*, &c., parallel to BT: then AB will be divided in the points, *p*, *q*, &c., into the required number of equal parts. The proof is manifest from the proposition in the text.



When the line to be divided is short, the following method is sometimes adopted.

Upon a line of any convenient length set as many equal spaces QP, PN, &c. as the given line is to be divided into: from Q as a centre, with a radius equal to the distance from Q to A (the further extremity of the last space), describe an arc, as AB; and in that arc place the straight line AB, equal to the given line. Join Q, B, and make Q*p*, Q*n*, &c., each equal to QP, QN, &c.; also join the points on QA and QB by the lines P*p*, N*n*, &c. The triangle AQB is similar (6. 6.) to each of the triangles PQ*p*, NQ*n*, &c.; therefore P*p*, N*n*, &c., are such parts of AB, the given line, as QP, QN, &c., are of QA: or P*p* is equal to one of the parts, N*n* is equal to two of the parts, &c. into which AB was to be divided.



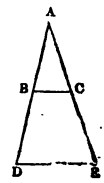
PROP. XI. PROB.

To find a third proportional to two given straight lines.

Let AB, AC be the two given straight lines, and let them be placed so as to contain any angle: it is required to find a third proportional to AB, AC.

Produce AB, AC to the points D, E; and make BD equal to AC; and having joined B, C, through D, draw DE parallel to it (32. 1.)

Because BC is parallel to DE, a side of the triangle ADE, AB is (2. 6.) to BD, as AC to CE: but BD is equal to AC; as therefore AB to AC, so is AC to CE. Wherefore, to the two given straight lines AB, AC, a third proportional, CE is found. Q. E. F.

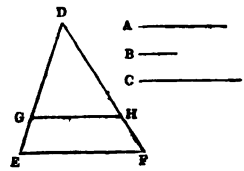


PROP. XII. PROB.

To find a fourth proportional to three given straight lines.

Let A, B, C be the three given straight lines: it is required to find a fourth proportional to A, B, C.

Take two straight lines DE, DF, containing any angle EDF; and upon these make DG equal to A, GE equal to B, and DH equal to C; and having joined G, H, draw EF parallel (31. 1.) to it through the point E: and because GH is parallel to EF, one of the sides of the triangle DEF, DG is to GE, as DH to HF (2. 6.) but DG is equal to A, GE to B, and DH to C; thereof as A is to B, so is C to HF. Wherefore to the three given straight lines A, B, C, a fourth proportional, HF is found. Q. E. F.

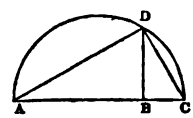


PROP. XIII. PROB.

To find a mean proportional between two given straight lines.

Let AB, BC be the two given straight lines: it is required to find a mean proportional between them.

Place AB, BC in a straight line, and upon AC describe the semicircle ADC, and from the point B draw (11. 1.) BD at right angles to AC, also draw AD, DC.



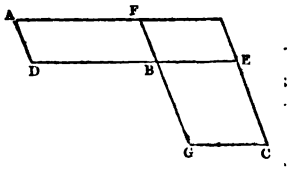
Because the angle ADC in a semicircle is a right angle (31. 3.), and because in the right-angled triangle ADC, BD is drawn from the right angle perpendicular to the base, DB is a mean proportional between AB, BC, the segments of the base (Cor. 8. 6.). Therefore between the two given straight lines AB, BC, a mean proportional, DB is found. Q. E. F.

PROP. XIV. THEOR.

EQUIVALENT parallelograms, which have one angle of the one equal to one angle of the other, have their sides about the equal angles reciprocally proportional: and parallelograms that have one angle of the one equal to one angle of the other, and their sides about the equal angles reciprocally proportional, are equivalent to one another.

Let AB, BC be equivalent parallelograms, which have the angles at B equal, and let the sides DB, BE be placed in the same straight line; wherefore also FB, BG are in one straight line (14. 1.) The sides of the parallelograms AB, BC about the equal angles are reciprocally proportional; that is, DB is to BE, as GB to BF.

Complete the parallelogram FE; and because the parallelogram AB is equivalent to BC, and that FE is another parallelogram, AB is to FE, as BC to FE (3. Cor. Def. Pr.). But as AB to FE, so is the base DB to BE (1. 6.); and as BC to FE, so is the base GB to BF: therefore, as DB to BE so is GB to BF (5 Cor. Def. Pr.). Wherefore, the sides of the parallelograms AB, BC about their equal angles are reciprocally proportional.



But, let the sides about the equal angles be reciprocally proportional, viz. as DB to BE, so GB to BF: the parallelogram AB is equivalent to the parallelogram BC.

Because as DB to BE, so is GB to BF; and as DB to BE, so is the parallelogram AB to the parallelogram FE; and as GB to BF, so is the parallelogram BC to the parallelogram FE; therefore as AB to FE, so BC to FE (5. Cor. Def. Pr.): wherefore the parallelogram AB is equivalent (4. Cor. Def. Pr.) to the parallelogram BC. Therefore equivalent parallelograms, &c. Q. E. D.

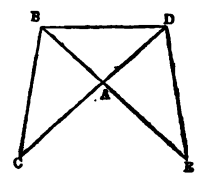
PROP. XV. THEOR.

EQUIVALENT triangles, which have one angle of the one equal to one angle of the other, have their sides about the equal angles reciprocally proportional: and triangles which have one angle in the one equal to one angle in the other, and their sides about the equal angles reciprocally proportional, are equivalent to one another.

Let ABC, ADE be equivalent triangles, which have the angle BAC equal to the angle DAE: the sides about the equal angles of the triangles are reciprocally proportional; that is, CA is to AD, as EA to AB.

Let the triangles be placed so that their sides CA, AD may be in one straight line; wherefore also EA and AB are in

one straight line (14. 1.); and join B, D. Because the triangle ABC is equal to the triangle ADE, and that ABD is another triangle; therefore as the triangle CAB is to the triangle BAD, so is triangle EAD to triangle DAB (3. Cor. Def. Pr.): but as triangle CAB to triangle BAD, so is the base CA to AD (1. 6.); and as triangle EAD to triangle DAB, so is the base EA to AB (1. 6.): as therefore CA to AD, so is EA to AB (5. Cor. Def. Pr.); wherefore the sides of the triangles ABC, ADE about the equal angles are reciprocally proportional.



But let the sides of the triangles ABC, ADE about the equal angles be reciprocally proportional, viz. CA to AD, as EA to AB: the triangle ABC is equivalent to the triangle ADE.

Having joined B, D as before; because, as CA to AD, so is EA to AB; and as CA to AD, so is triangle ABC to triangle BAD (1. 6.); and as EA to AB, so is triangle EAD to triangle BAD (1. 6.); therefore (5. Cor. Def. Pr.) as triangle BAC to triangle BAD, so is triangle EAD to triangle BAD; that is, the triangles BAC, EAD have the same ratio to the triangle BAD: wherefore the triangle ABC is equivalent (4. Cor. Def. Pr.) to the triangle ADE. Therefore equivalent triangles, &c. Q. E. D.

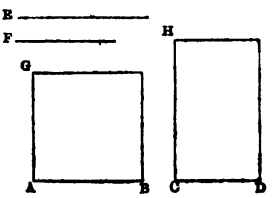
PROP. XVI. THEOR.

If four straight lines be proportionals, the rectangle contained by the extremes is equivalent to the rectangle contained by the means: and if the rectangle contained by the extremes be equivalent to the rectangle contained by the means, the four straight lines are proportionals.

Let the four straight lines AB, CD, E, F be proportionals, viz. as AB to CD, so E to F: the rectangle contained by AB, F is equivalent to the rectangle contained by CD, E.

From the points A, C draw (11. 1.) AG, CH at right angles to AB, CD; and make AG equal to F, and CH equal to E, and complete the parallelograms BG, DH. Because, as AB to CD, so is E to F; and that E is equal to CH, and F to AG; AB is (3. Cor. Def. Pr.) to CD, as CH to AG: therefore the sides of the parallelograms BG, DH about the equal angles are reciprocally proportional; but parallelograms which have their

sides about equal angles reciprocally proportional, are equivalent to one another (14. 6.); therefore the parallelogram BG is equivalent to the parallelogram DH: and the parallelogram BG is contained by the straight lines AB, F; because AG is equal to F; and the parallelogram DH is contained by CD and E; because CH is equal to E: therefore the rectangle contained by the straight lines AB, F is equivalent to that which is contained by CD and E.



And if the rectangle contained by the straight lines AB, F be equivalent to that which is contained by CD, E; these four lines are proportional, viz. AB is to CD, as E to F.

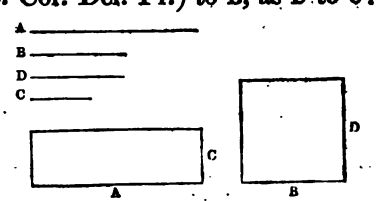
The same construction being made, because the rectangle contained by the straight lines AB, F is equivalent to that which is contained by CD, E, and that the rectangle BG is contained by AB, F, because AG is equal to F; and the rectangle DH by CD, E, because CH is equal to E; therefore the parallelogram BG is equivalent to the parallelogram DH; and they are equiangular: but the sides about the equal angles of equal parallelograms are reciprocally proportional (14. 6.): wherefore, as AB to CD, so is CH to AG; and CH is equal to E, and AG to F: as therefore AB is to CD, so is E to F. Wherefore, if four, &c. Q. E. D.

PROP. XVII. THEOR.

If three straight lines be proportionals, the rectangle contained by the extremes is equivalent to the square of the mean: and if the rectangle contained by the extremes be equivalent to the square of the mean, the three straight lines are proportionals.

Let the three straight lines, A, B, C be proportionals, viz. as A to B, so B to C: the rectangle contained by A, C is equivalent to the square of B.

Take D equal to B, and because as A to B, so B to C, and that B is equal to D; A is (3. Cor. Def. Pr.) to B, as D to C: but if four straight lines be proportionals, the rectangle contained by the extremes is equivalent to that which is contained by the means (16. 6.): therefore the rectangle contained by A, C is



equivalent to that contained by B, D: but the rectangle contained by B, D is the square of B; because B is equal to D: therefore the rectangle contained by A, C is equivalent to the square of B.

And if the rectangle contained by A, C be equivalent to the square of B: A is to B, as B to C.

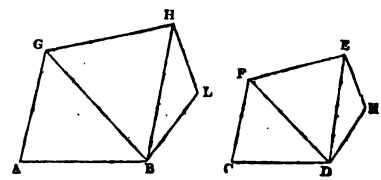
The same construction being made, because the rectangle contained by A, C is equivalent to the square of B, and the square of B is equivalent to the rectangle contained by B, D, because B is equal to D; therefore the rectangle contained by A, C is equivalent to that contained by B, D; but if the rectangle contained by the extremes be equivalent to that contained by the means, the four straight lines are proportionals (16. 6.); therefore A is to B, as D to C; but B is equal to D; wherefore, as A to B, so B to C: therefore, if three straight lines, &c. Q. E. D.

PROP. XVIII. PROB.

UPON a given straight line to describe a rectilineal figure, similar, and similarly situated to a given rectilineal figure.

Let AB be the given straight line, and CDEF the given rectilineal figure of four sides: it is required upon the given straight line AB to describe a rectilineal figure similar, and similarly situated to CDEF.

Join D, F, and at the points A, B, in the straight line AB, make (23. 1.) the angle BAG equal to the angle at C, and the angle ABG equal to the angle CDF; therefore the remaining angle CFD is equal to the remaining angle AGB (32. 1.): wherefore the triangle FCD is equiangular to the triangle GAB. Again, at the points G, B, in the straight line GB, make (23. 1.) the angle BGH equal to the angle DFE, and the angle GBH equal to FDE: therefore the remaining angle FED is equal to the remaining angle GHB, and the triangle FDE equiangular to the triangle GBH: then, because the angle AGB is equal to the angle CFD, and BGH to DFE, the whole angle AGH is equal to the whole CFE. For the same reason, the angle ABH is equal to the angle CDE; also the angle at A is equal to the angle at C, and the angle GHB to FED: therefore the rectilineal figure ABHG is equiangular to CDEF. But like-



wise these figures have their sides about the equal angles proportionals; because the triangles GAB, FCD being equiangular, BA is (4. 6.) to AG, as DC to CF; and because AG is to GB, as CF to FD; and as GB to GH, so (by reason of the equiangular triangles BGH, DFE) is FD to FE; therefore, ex æquali (10. Pr.), AG is to GH, as CF to FE. In the same manner it may be proved that AB is to BH, as CD to DE: and GH is to HB, as FE to ED (4. 6.): wherefore, because the rectilineal figures ABHG, CDEF are equiangular, and have their sides about the equal angles proportionals, they are similar to one another (1. Def. 6.).

Next, let it be required to describe upon a given straight line AB, a rectilineal figure similar, and similarly situated to the rectilineal figure CDKEF.

Join D, E, and upon the given straight line AB describe the rectilineal figure ABHG similar, and similarly situated to the quadrilateral figure CDEF, by the former case; and at the points B, H, in the straight line BH, make the angle HBL equal to the angle EDK, and the angle BHL equal to the angle DEK; therefore the remaining angle at K is equal to the remaining angle at L: and because the figures ABHG, CDEF are similar, the angle GHB is equal to the angle FED, and BHL is equal to DEK; wherefore the whole angle GHL is equal to the whole angle FEK. For the same reason, the angle ABL is equal to the angle CDK: therefore the five-sided figures AGHLB, CFEKD are equiangular; and because the figures AGHB, CFED are similar, GH is to HB, as FE to ED; and as HB to HL, so is ED to EK (4. 6.); therefore, ex æquali (10. Pr.), GH is to HL, as FE to EK. For the same reason, AB is to BL, as CD to DK: and BL is to LH, as (4. 6.) DK to KE, because the triangles BLH, DKE are equiangular: therefore because the five-sided figures AGHLB, CFEKD are equiangular, and have their sides about the equal angles proportionals, they are similar to one another; and in the same manner a rectilineal figure of six or more sides may be described upon a given straight line similar to one given, and so on. Q. E. F.

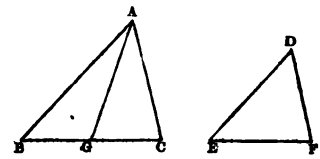
PROP. XIX. THEOR.

SIMILAR triangles are to one another in the duplicate ratio of their homologous sides.

Let ABC, DEF be similar triangles, having the angle B equal to the angle E, and let AB be to BC, as DE to EF, so that the side BC is homologous to EF (13. Def. Pr.): the triangle

ABC has to the triangle DEF the duplicate ratio of that which BC has to EF.

Take BG a third proportional to BC, EF (11. 6.): so that BC is to EF, as EF to BG, and join G, A: then, because, as AB to BC, so DE to EF; alternately (4. Pr.), AB is to DE, as BC to EF: but as BC to EF, so is EF to BG; therefore (5. Cor. Def. Pr.) as AB to DE, so is EF to BG; wherefore the sides of the triangles ABG, DEF which are about the equal angles are reciprocally proportional: but triangles which have the sides about two equal angles reciprocally proportional are equivalent to one another (15. 6.): therefore the triangle ABG is equivalent to the triangle DEF: and because as BC is to EF, so EF to BG; and that if three straight lines be proportionals, the first is said (17. Def. Pr.) to have to the third the duplicate ratio of that which it has to the second; BC therefore has to BG the duplicate ratio of that which BC has to EF: but as BC to BG, so is (1. 6.) the triangle ABC to the triangle ABG. Therefore the triangle ABC has to the triangle ABG the duplicate ratio of that which BC has to EF: but the triangle ABG is equivalent to the triangle DEF: wherefore also the triangle ABC has to the triangle DEF the duplicate ratio of that which BC has to EF. Therefore similar triangles, &c. Q. E. D.



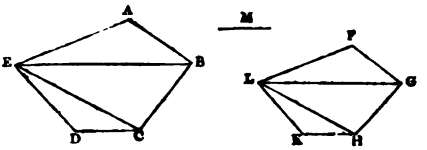
Cor. From this it is manifest that if three straight lines be proportionals, as the first is to the third, so is any triangle upon the first to a similar, and similarly described triangle upon the second.

PROP. XX. THEOR.

Similar polygons may be divided into the same number of similar triangles, having the same ratio to one another that the polygons have; and the polygons have to one another the duplicate ratio of that which their homologous sides have.

Let ABCDE, FGHKL be similar polygons, and let AB be the homologous side to FG: the polygons ABCDE, FGHKL may be divided into the same number of similar triangles, of which each to each has the same ratio which the polygons have; and the polygon ABCDE has to the polygon FGHKL the duplicate ratio of that which the side AB has to the side FG.

Draw BE, EC, GL, LH: and because the polygon ABCDE is similar to the polygon FGHKL, the angle BAE is equal to the angle GFL (1. Def. 6.) and BA is to AE, as GF to FL (1. Def. 6.): wherefore, because the triangles ABE, FGL have an angle in one equal to an angle in the other, and their sides about these equal angles proportionals, the triangle ABE is equiangular (6. 6.), and therefore similar to the triangle FGL (4. 6.); wherefore the angle ABE is equal to the angle FGL; and, because the polygons are similar, the whole angle ABC is equal (1. Def. 6.) to the whole angle FGH; therefore the remaining angle EBC is equal to the remaining angle LGH: and because the triangles ABE, FGL are similar, EB is to BA, as LG to GF (1. Def. 6.); and also, because the polygons are similar, AB is to BC, as FG to GH (1. Def. 6.); therefore, ex æquali (10. Pr.), EB is to BC, as LG to GH; that is, the sides about the equal angles EBC, LGH are proportionals; therefore the triangle EBC is equiangular to the triangle LGH and similar to it (4. 6.). For the same reason, the triangle ECD likewise is similar to the triangle LHK: therefore the similar polygons ABCDE, FGHKL are divided into the same number of similar triangles.



Also these triangles have, each to each, the same ratio which the polygons have to one another, the antecedents being ABE, EBC, ECD, and the consequents FGL, LGH, LHK: and the polygon ABCDE has to the polygon FGHKL the duplicate ratio of that which the side AB has to the homologous side FG.

Because the triangle ABE is similar to the triangle FGL, ABE has to FGL the duplicate ratio (19. 6.) of that which the side BE has to the side GL: for the same reason, the triangle BEC has to GLH the duplicate ratio of that which BE has to GL: therefore, as the triangle ABE to the triangle FGL, so (5. Cor. Def. Pr.) is the triangle BEC to the triangle GLH. Again, because the triangle EBC is similar to the triangle LGH, EBC has to LGH the duplicate ratio of that which the side EC has to the side LH. For the same reason, the triangle ECD has to the triangle LHK the duplicate ratio of that which EC has LH: as therefore the triangle EBC to the triangle LGH, so is (5. Cor. Def. Pr.) the triangle ECD to the triangle LHK: but it has been proved that the triangle EBC is likewise to the triangle LGH, as the triangle ABE to

the triangle FGL: therefore, as the triangle ABE to the triangle FGL, so is triangle EBC to triangle LGH, and triangle ECD to triangle LHK: and therefore, as one of the antecedents to one of the consequents, so are all the antecedents to all the consequents (9. Pr.). Wherefore, as the triangle ABE to the triangle FGL, so is the polygon ABCDE to the polygon FGHKL: but the triangle ABE has to the triangle FGL the duplicate ratio of that which the side AB has to the homologous side FG. Therefore also the polygon ABCDE has to the polygon FGHKL the duplicate ratio of that which AB has to the homologous side FG. Wherefore similar polygons, &c. Q. E. D.

COR. 1. In like manner it may be proved that similar figures of four, or of any number of sides, are one to another in the duplicate ratio of their homologous sides, and it has already been proved in triangles. Therefore, universally, similar rectilineal figures are to one another in the duplicate ratio of their homologous sides.

N. B. Squares are similar figures, and therefore they are to one another in the duplicate ratio of their sides: hence, similar plane figures are to one another as the squares of their homologous sides. The square of a line as AB, is, for the sake of brevity, represented by AB^2.

COR. 2. And if to AB, FG, two of the homologous sides, a third proportional M be taken, AB has (17. Def. Pr.) to M the duplicate ratio of that which AB has to FG; but the four-sided figure or polygon upon AB has to the four-sided figure or polygon upon FG likewise the duplicate ratio of that which AB has to FG: therefore, as AB is to M, so is the figure upon AB to the figure upon FG, which was also proved in triangles (Cor. 19. 6.). Therefore, universally, it is manifest that if three straight lines be proportionals, as the first is to the third, so is any rectilineal figure upon the first, to a similar, and similarly described rectilineal figure upon the second.

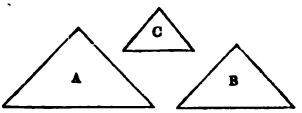
PROP. XXI. THEOR.

RECTILINEAL figures of any kind, which are similar to the same rectilineal figure, are also similar to one another.

Let each of the rectilineal figures A, B, which may be of any kind, be similar to the rectilineal figure C: the figure A is similar to the figure B.

Because A is similar to C, they are equiangular, and also have their sides about the equal angles proportionals

(1. Def. 6.). Again, because B is similar to C, they are equiangular, and have their sides about the equal angles proportionals (1. Def. 6.). Therefore the figures A, B are each of them equiangular to C, and have the sides about the equal angles of each of them and of C proportionals. Wherefore the rectilineal figures A and B are equiangular (1. Ax. 1.), and have their sides about the equal angles proportionals (4. 6.). Therefore A is similar to B. Q. E. D.

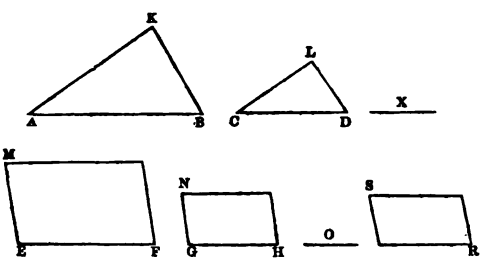


PROP. XXII. THEOR.

If four straight lines be proportionals, the similar rectilineal figures similarly described upon them shall also be proportionals: and if the similar rectilineal figures similarly described upon four straight lines be proportionals, those straight lines shall be proportionals.

Let the four straight lines AB, CD, EF, GH be proportionals, viz. AB to CD, as EF to GH, and upon AB, CD let the similar rectilineal figures KAB, LCD be similarly described; and upon EF, GH the similar rectilineal figures MF, NH, in like manner: the rectilineal figure KAB is to LCD, as MF to NH.

To AB, CD take a third proportional (11. 6.) X, and to EF, GH a third proportional O: and because AB is to CD, as EF to GH, and that CD is (5. Cor. Def. Pr.) to X, as GH to O; wherefore, ex æquali (10. Pr.), as AB to X, so EF to O: but as AB to X, so is (2. Cor. 20. 6.) the rectilineal KAB to the rectilineal LCD, and as EF to O, so is (2. Cor. 20. 6.) the rectilineal MF to the rectilineal NH: therefore, as KAB to LCD, so (5. Cor. Def. Pr.) is MF to NH.



And if the rectilineal KAB be to LCD, as MF to NH; the straight line AB is to CD, as EF to GH.

Make (12. 6.) as AB to CD, so EF to PR, and upon PR de-

scribe (18. 6.) the rectilineal figure SR similar, and similarly situated to either of the figures MF, NH: then, because as AB to CD, so is EF to PR, and that upon AB, CD are described the similar, and similarly situated rectilineals KAB, LCD, and upon EF, PR in like manner the similar rectilineals MF, SR; KAB is to LCD, as MF to SR; but by the hypothesis KAB is to LCD, as MF to NH; and therefore the rectilineal MF having the same ratio to each of the two NH, SR, these are equal (4. Cor. Def. Pr.) to one another: they are also similar, and similarly situated; therefore GH is equal to PR: and because as AB to CD, so is EF to PR, and that PR is equal to GH; AB is to CD, as EF to GH. If therefore four straight lines, &c. Q. E. D.

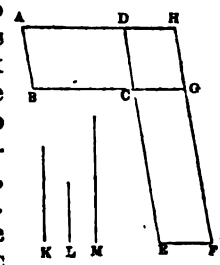
N. B. Squares being similar figures, if four straight lines be proportionals, the squares described upon them are proportionals: that is, if A, B, C, D be four straight lines, and if A : B :: C : D, then $A^2 : B^2 :: C^2 : D^2$.

PROP. XXIII. THEOR.

EQUIANGULAR parallelograms have to one another the ratio which is compounded of the ratio of their sides.

Let AC, CF be equiangular parallelograms, having the angle BCD equal to the angle ECG: the ratio of the parallelogram AC to the parallelogram CF is the same as the ratio which is compounded of the ratios of their sides.

Let BC, CG be placed in a straight line; therefore DC and CE are also in a straight line (14. 1.), and complete the parallelogram DG; and taking any straight line K, make (12. 6.) as BC to CG, so K to L; and as DC to CE, so make L to M: therefore, the ratios of K to L, and L to M, are the same as the ratios of the sides, viz. of BC to CG, and DC to CE. But the ratio of K to M is that which is said to be compounded (16. Def. Pr.) of the ratios of K to L, and L to M; wherefore also K has to M the ratio compounded of the ratios of the sides: and because as BC to CG, so is the parallelogram AC to the parallelogram CH (1. 6.); but as BC to CG, so is K to L; therefore K is (5. Cor. Def. Pr.) to L, as the parallelogram AC to the parallelogram CH. Again, because as DC to CE, so is the parallelogram CH to the parallelogram CF; but as DC to CE, so is L to M; wherefore L is (5. Cor. Def.



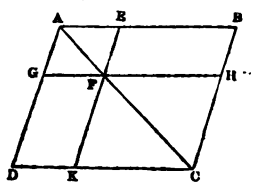
Pr.) to M, as the parallelogram CH to the parallelogram CF: therefore, since it has been proved, that as K to L, so is the parallelogram AC to the parallelogram CH; and as L to M, so the parallelogram CH to the parallelogram CF; ex æquali (10. Pr.), K is to M, as the parallelogram AC to the parallelogram CF: but K has to M the ratio which is compounded of the ratios of the sides; therefore also the parallelogram AC has to the parallelogram CF the ratio which is compounded of the ratios of the sides. Wherefore equiangular parallelograms, &c. Q. E. D.

PROP. XXIV. THEOR.

THE parallelograms about the diameter of any parallelogram are similar to the whole, and to one another.

Let ABCD be a parallelogram, of which the diameter is AC; and EG, HK the parallelograms about the diameter: the parallelograms EG, HK are similar both to the whole parallelogram ABCD, and to one another.

Because DC, GF are parallels, the angle ADC is equal (29. 1.) to the angle AGF: for the same reason, because BC, EF are parallels, the angle ABC is equal to the angle AEF: and each of the angles BCD, EFG is equal to the opposite angle DAB (34. 1.), and therefore are equal to one another; wherefore the parallelograms ABCD, AEFG are equiangular: and because the angle ABC is equal to the angle AEF, and the angle BAC common to the two triangles BAC, EAF, they are equiangular to one another; therefore (4. 6.) as AB to BC, so is AE to EF. And because the opposite sides of parallelograms are equal to one another (34. 1.), AB is to AD, as AE to AG (3. Cor. Def. Pr.); and DC to CB, as GF to FE; and also CD to DA, as FG to GA: therefore the sides of the parallelograms ABCD, AEFG about the equal angles are proportionals; and they are therefore similar to one another (1. Def. 6.): for the same reason the parallelogram ABCD is similar to the parallelogram FHCK. Wherefore each of the parallelograms GE, KH is similar to DB: but rectilineal figures which are similar to the same rectilineal figure are also similar to one another (21. 6.); therefore the parallelogram GE is similar to KH. Wherefore the parallelograms, &c. Q. E. D.

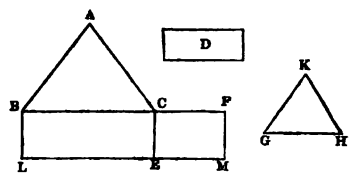


PROP. XXV. PROB.

To describe a rectilineal figure which shall be similar to one, and equivalent to another given rectilineal figure.

Let ABC be the given rectilineal figure, to which the figure to be described is required to be similar, and D that to which it must be equivalent: it is required to describe a rectilineal figure similar to ABC, and equal to D.

Upon the straight line BC describe (Cor. 45. 1.) the parallelogram BE equivalent to the figure ABC: also upon CE describe (Cor. 45. 1.) the parallelogram CM equivalent to D, and having the angle FCE equal to the angle CBL: therefore BC and CF are in a straight line (29. 1. 14. 1.), as also LE and EM: between BC and CF find (13. 6.) a mean proportional GH, and upon GH describe (18. 6.) the rectilineal figure KGH, similar, and similarly situated to the figure ABC. And because BC is to GH, as GH to CF, and if three straight lines be proportionals, as the first is to the third, so is (2. Cor. 20. 6.) the figure upon the first to the similar, and similarly described figure upon the second; therefore, as BC to CF, so is the rectilineal figure ABC to KGH: but as BC to CF, so is (1. 6.) the parallelogram BE to the parallelogram EF: therefore, as the rectilineal figure ABC is to KGH, so is the parallelogram BE to the parallelogram EF (5. Cor. Def. Pr.); and the rectilineal figure ABC is equivalent to the parallelogram BE; therefore the rectilineal figure KGH is equivalent (4. Cor. Def. Pr.) to the parallelogram EF: but EF is equivalent to the figure D; wherefore also KGH is equivalent to D; and it is similar to ABC. Therefore the rectilineal figure KGH has been described similar to the figure ABC, and equivalent to D. Q. E. F.

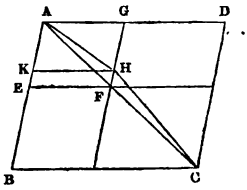


PROP. XXVI. THEOR.

If two similar parallelograms have a common angle, and be similarly situated, they are about the same diameter.

Let the parallelograms ABCD, AEFG be similar, and similarly situated, and have the angle DAB common: ABCD and AEFG are about the same diameter.

For if not, let, if possible, the parallelogram BD have its diameter AHC in a different straight line from AF, the diameter of the parallelogram EG, and let GF meet AHC in H; and through H draw HK parallel to AD, or BC: therefore the parallelograms ABCD, AKHG being about the same diameter, they are similar to one another (24. 6.): wherefore as DA to AB, so is (1. Def. 6.) GA to AK. But because ABCD and AEFG are similar parallelograms, as DA is to AB, so is GA to AE; therefore (5. Cor. Def. Pr.) as GA to AE, so GA to AK: wherefore GA has the same ratio to each of the straight lines AE, AK; and consequently AK is equal (4. Cor. Def. Pr.) to AE, the less to the greater, which is impossible: therefore ABCD and AKHG are not about the same diameter; wherefore ABCD and AEFG must be about the same diameter. Therefore, if two similar, &c. Q. E. D.

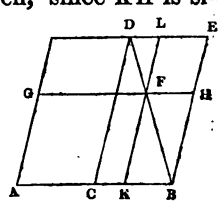


PROP. XXVII. THEOR.

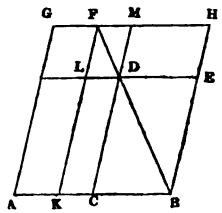
If there be applied on a given line equiangular parallelograms, each differing, in defect, from a parallelogram standing on the whole given line, and having the same altitude, by a parallelogram similar to one which is formed on half the given line: that parallelogram which stands on half the line is the greatest.

Let AB be the given line, and AD, CE be parallelograms standing on AC and CB, each of which is the half of AB; also let AF be any parallelogram equiangular to AD and differing, in defect, from AH by the parallelogram KH, which is similar to AD, or CE: AD shall be greater than any parallelogram, AF.

First, let AK be greater than AC: then, since KH is similar to CE, their diagonals FB and DB coincide (26. 6.) in direction, and (43. 1.) CF is equal to FE; therefore CH, or CG, is equal to KE: adding CF to both, AF is equal to the gnomon CHL: but CHL is less than CE, or AD; therefore, &c.



Next, let AK be less than AC: then, since DH is equal to DG (36. 1.), DH is greater than LG: but (43. 1.) DH is equal to DK; therefore DK is greater than LG, and adding AL to both, AD will be greater than AF; therefore, &c. Q. E. D.



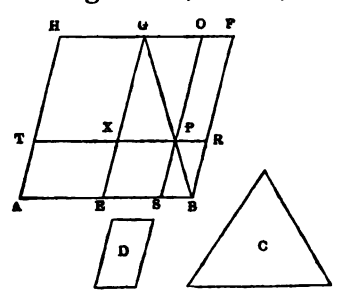
PROP. XXVIII. PROB.

On a given line to apply a parallelogram equal to a given rectilineal figure, and differing, in defect, from a parallelogram standing on the whole given line, and having the same altitude, by a parallelogram similar to a given parallelogram.

[Note. — The given rectilineal figure must not be greater than a parallelogram standing on half the given line, and similar to the given parallelogram.]

Let AB be the given line, C the given rectilineal figure to which the required parallelogram is to be equal, and let D be the given parallelogram.

Bisect AB in E; form the parallelogram EF, similar, and similarly situated to D, (18.6.) and complete the parallelogram AG: this last will, by the conditions, either be equal to C or greater than C. If AG be equal to C, the thing required is done, for AG differs from AF, in defect, by EF, which is similar to D. If not, since C is less than EF, there may be formed on GE and GF (25. 6.) a parallelogram XO, similar to EF, or to D, and equal to the excess of EF above C; then (26. 6.) XO, EF will be about the same diameter, and SR will be similar to XO; that is, to D: also the gnomon ERO will be equal to C. But (43. 1.) OR is equal to XS, and XB, or AX, to OB; adding XS to OB and AX, AP will be equal to the gnomon ERO; that is, to the figure C. Thus AP is a parallelogram applied on AB, equal to C, and differing, in defect, from AR (on AB) by a parallelogram SR, which is similar to D. Q. E. F.

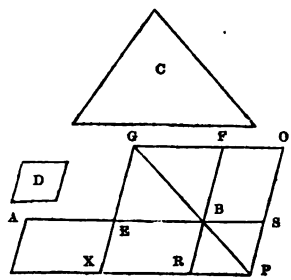


PROP. XXIX. PROB.

On a given line to apply a parallelogram equal to a given rectilineal figure, and differing, in excess, from a parallelogram formed on the whole line, and having the same altitude, by a parallelogram similar to a given parallelogram.

Let AB be the given line, C the given rectilineal figure to which the required parallelogram is to be equal, and let D be the given parallelogram.

Bisect AB in E; form the parallelogram EF, similar, and similarly situated to D (18.6.). On GE and GF produced make a parallelogram XO, similar to EF, or D, and equal to the sum of the figures EF and C; and complete the parallelogram AX: then (26. 6.) XO, EF will be about the same diameter, and SR will be similar to XO, that is, to D; also the gnomon ERO will be equal to C. But (43. 1.) OB is equal to XB, or AX; and adding EP to OB and AX, AP will be equal to the gnomon ERO; that is, to the figure C. Thus AP is a parallelogram applied on AB, equal to C, and differing, in excess, from AR (on AB) by a parallelogram SR, which is similar to D. Q. E. F.

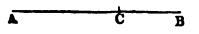


PROP. XXX. PROB.

To cut a given straight line in extreme and mean ratio.

Let AB be the given straight line: it is required to cut it in extreme and mean ratio.

Divide AB in the point C, so that the rectangle contained by AB, BC may be equivalent to the square of AC (11.2.). Then, because the rectangle AB, BC is equivalent to the square of AC, as BA is to AC, so is AC to CB (17. 6.). Therefore (3. Def. 6.) AB is cut in extreme and mean ratio in C. Q. E. F.



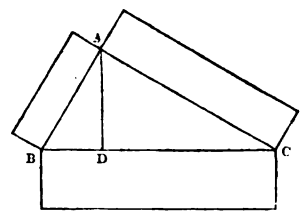
PROP. XXXI. THEOR.

In right-angled triangles, the rectilineal figure described upon the side opposite to the right angle, is

equivalent to the similar, and similarly described figure upon the sides containing the right angle.

Let ABC be a right-angled triangle, having the right angle BAC: the rectilineal figure described upon BC shall be equivalent to the similar, and similarly described figures upon BA, AC.

Draw the perpendicular AD; therefore, because in the right-angled triangle ABC, AD is drawn from the right angle at A perpendicular to the base BC, the triangles ABD, ADC are similar to the whole triangle ABC, and to one another (8. 6.); and because the triangle ABC is similar to ADB, as CB to BA, so is BA to BD (4. 6.); and because these three straight lines are proportionals, as the first to the third, so is the figure upon the first to the similar, and similarly described figure upon the second (2 Cor. 20. 6.). Therefore as CB to BD, so is the figure upon CB to the similar, and similarly described figure upon BA: and inversely (3 Pr.) as DB to BC, so is the figure upon BA to that upon BC. For the same reason, as DC to CB, so is the figure upon CA to that upon CB. Wherefore, as BD and DC together to BC, so are the figures upon BA, AC to that upon BC (8 Pr.): but BD and DC together are equal to BC: therefore the figure described on BC is equivalent (2. Cor. Def. Pr.) to the similar, and similarly described figures on BA, AC. Wherefore, in right-angled triangles, &c. Q. E. D.

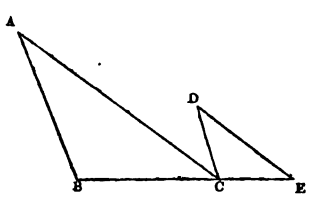


PROP. XXXII. THEOR.

If two triangles, which have two sides of the one proportional to two sides of the other, be joined at one angle, so as to have their homologous sides parallel to one another: the remaining sides shall be in a straight line.

Let ABC, DCE be two triangles which have the two sides BA, AC proportional to the two CD, DE, viz. BA to AC, as CD to DE; and let AB be parallel to DC, and AC to DE: BC and CE are in a straight line.

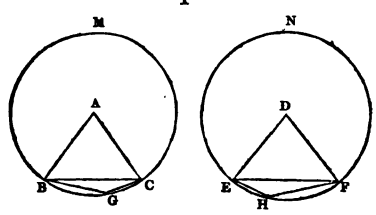
Because AB is parallel to DC, and the straight line AC



meets them, the alternate angles BAC, ACD are equal (29. 1.); for the same reason, the angle CDE is equal to the angle ACD; wherefore also BAC is equal to CDE. And because the triangles ABC, DCE have one angle at A equal to one at D, and the sides about these angles proportionals, viz. BA to AC, as CD to DE, the triangle ABC is equiangular (6. 6.) to DCE: therefore the angle ABC is equal to the angle DCE; and the angle BAC was proved to be equal to ACD; therefore the whole angle ACE is equal to the two angles ABC, BAC: add the common angle ACB; then the angles ACE, ACB are equal to the angles ABC, BAC, ACB. But ABC, BAC, ACB are equal to two right angles (32. 1.); therefore also the angles ACE, ACB are equal to two right angles. And since at the point C, in the straight line AC, the two straight lines BC, CE, which are on the opposite sides of it, make the adjacent angles ACE, ACB equal to two right angles; therefore (14. 1.) BC and CE are in a straight line. Wherefore, if two triangles, &c. Q. E. D.

PROP. AA. THEOR.

In the same or equal circles, sectors which stand upon equal circumferences are equal to one another.



Let the sectors BAC and EDF of the equal circles BMC and ENF, stand upon the equal arcs BC and EF: they are equal to one another.

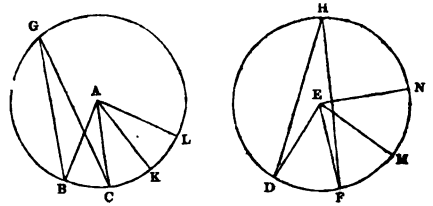
In the arcs BC and EF take any points G and H, and draw BG, GC, BC, EH, HF, EF. Because the arcs BC and EF are equal, the angles BAC and EDF are equal (27. 3.) But BA and AC are equal to ED and DF; therefore the triangle BAC is equal to the triangle EDF, and the base BC to the base EF (4. 1.). And because the arcs BC and EF are equal, the arcs BMC and ENF, which are the remainders of the whole circumferences, are likewise equal; wherefore the angle BGC is equal to the angle EHF (27. 3.): the segments BGC and EHF are therefore similar (11. Def. 3.); and they are upon the equal straight lines BC and EF, therefore they are equal to one an-

other (24. 3.). And because the triangle BAC is equal to the triangle EDF, and the segment BGC to the segment EHF, the sector BAC is equal to the sector EDF. Q. E. D.

PROP. XXXIII. THEOR.

In equal circles, angles, whether at the centres or circumferences, have the same ratio to one another as the arcs have on which they stand: so also have the sectors.

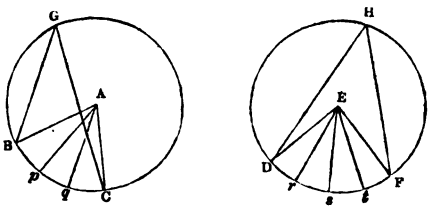
Let BGC and DHF be equal circles; and let BAC, DEF be angles at their centres; also let BGC and DHF be angles at their circumferences; as the arc BC is to the arc DF, so is the angle BAC to the angle DEF, and the angle BGC to the angle DHF.



Take any number of arcs CK, KL each equal to BC, and any number whatever FM, MN each equal to DF: and draw AK, AL, EM, EN. Because the arcs BC, CK, KL are all equal, the angles BAC, CAK, KAL are all also equal (27. 3.); therefore what multiple soever the arc BL is of the arc BC, the same multiple is the angle BAL of the angle BAC. For the same reason, whatever multiple the arc DN is of the arc DF, the same multiple is the angle DEN of the angle DEF: and if the arc BL be equal to the circumference DN, the angle BAL is also equal to the angle DEN; and if the arc BL be greater than DN, likewise the angle BAL is greater than DEN; and if less, less. There being then four magnitudes, the two arcs BC, DF and the two angles BAC, DEF; of the arc BC and of the angle BAC have been taken any equimultiples whatever, viz. the arc BL and the angle BAL; and of the arc DF, and of the angle DEF, any equimultiples whatever, viz. the arc DN and the angle DEN. And it has been proved that if the arc BL be greater than DN, the angle BAL is greater than DEN; if equal, equal; and if less, less: as therefore the arc BC is to the arc DF, so is the angle BAC to the angle DEF (5 Def. 5. Euclid, or 10 Def. Pr.) But as the angle BAC is to the angle DEF so is the angle BGC to the angle DHF (1 Pr.); for each is double of each (20. 3.); therefore as the arc BC is to the arc DF, so is the angle BGC to the angle DHF.

Also as the arc BC is to the arc DF, so is the sector BAC to the sector DEF. For, by Prop. AA, the sectors BAC, CAK, KAL, standing upon equal arcs in the circle BGL, are all equal to one another; so likewise are the sectors DEF, FEM, MEN, in the circle DHN; therefore, reasoning precisely as before, but substituting the sectors for the angles at the centres of the circles, this part of the proposition will be proved. Wherefore in equal circles, &c. Q. E. D.

Otherwise, if the arcs BC and DF have a common measure, each of them may be divided into parts equal to that common measure. Let B*p*, *pq*, *q*C be the parts into which BC is divided; and D*r*, *rs*, *st*, *t*F the parts into which DF is divided:



draw A*p*, A*q*, E*r*, E*s*, E*t*. Then the angles BA*p*, *p*A*q*, &c. DE*r*, *r*E*s*, &c. are all equal to one another, because they stand upon equal arcs (27. 3.). Wherefore the arc BC contains the arc B*p*, as often as the angle BAC contains the angle BA*p*; and the arc DF contains the arc B*p*, as often as the angle DEF contains the angle BA*p*. The arc B*p* and the angle BA*p* are therefore like parts of the arc DF and of the angle DEF, and those parts are contained the same number of times in the arc BC and in the angle BAC: wherefore

BC : DF :: BAC : DEF (10. Def. Pr.).

Now the angle BGC is half of BAC, and the angle DHF half of DEF (20. 3.); and halves have the same ratio as their wholes (Cor. 1. Pr.); therefore

BGC : DHF :: BAC : DEF,

but BAC : DEF :: BC : DF,

therefore, by equality of ratios,

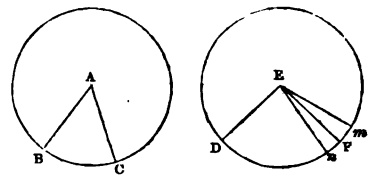
BGC : DHF :: BC : DF.

Also, as the arc BC to the arc DF, so is the sector BAC to the sector DEF. For, the same construction being made, the sectors BA*p*, *p*A*q*, &c. DE*r*, *r*E*s*, &c. are all equal to one another (AA. 6.). It is therefore plain that the arc B*p* and the sector BA*p* are like parts of the arc DF and of the sector DEF; and likewise that those parts are contained the same number of times in the arc BC and in the sector BAC: wherefore

BC : DF :: sector BAC : sector DEF (10. Def. Pr.)

If the arcs BC and DF are incommensurable, by continually bisecting one of them, as BC, there may at length

(D. Pr.) be found an arc equal to, or less than the least that can be assigned; this arc will be contained in DF a certain number of times with a remainder, as nF, or an excess, as Fm less than itself; that is, less than any arc that can be assigned. Join E, n and E, m; then, the arcs BC, Dn and BC, Dm being commensurable,



angle BAC : angle DEn :: BC : Dn

and angle BAC : angle DEm :: BC : Dm;

also the difference between Dn and Dm is equal to, or less than the least arc that can be assigned; therefore, by the Second General Theorem, (Proportion,)

angle BAC : angle DEF :: BC : DF.

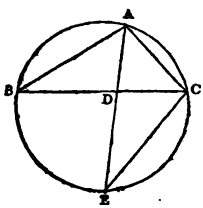
The same reasoning applies to sectors.

PROP. B. THEOR.

If an angle of a triangle be bisected by a straight line, which likewise cuts the base: the rectangle contained by the sides of the triangle is equivalent to the rectangle contained by the segments of the base, together with the square of the straight line bisecting the angle.

Let ABC be a triangle, and let the angle BAC be bisected by the straight line AD: the rectangle BA, AC is equivalent to the rectangle BD, DC, together with the square of AD.

Describe the circle (5. 4.) ACB about the triangle, and produce AD to the circumference in E, and join E, C. Then because the angle BAD is equal to the angle CAE, and the angle ABD to the angle (21. 3.) AEC, for they are in the same segment; the triangles ABD, AEC are equiangular to one another: therefore as BA to AD, so is (4. 6.) EA to AC, and consequently the rectangle BA, AC is equivalent (16. 6.) to the rectangle EA, AD, that is (3. 2.), to the rectangle ED, DA, together with the square of AD: but the rectangle ED, DA is equivalent to the rectangle (35. 3.) BD, DC; therefore the rectangle BA, AC is equivalent to the rectangle BD, DC, together with the square of AD. Wherefore, if an angle, &c. Q. E. D.

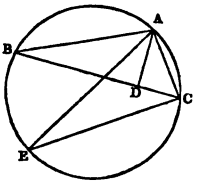


PROP. C. THEOR.

If from any angle of a triangle a straight line be drawn perpendicular to the base: the rectangle contained by the sides of the triangle is equivalent to the rectangle contained by the perpendicular and the diameter of the circle described about the triangle.

Let ABC be a triangle, and AD the perpendicular from the angle A to the base BC; the rectangle BA, AC is equivalent to the rectangle contained by AD, and the diameter of the circle described about the triangle.

Describe (5. 4.) the circle ACB about the triangle, and draw its diameter AE, and join E, C. Because the right angle BDA is equal (31. 3.) to the angle ECA in a semicircle, and the angle ABD to the angle AEC in the same segment (21. 3.); the triangles ABD, AEC are equiangular: therefore as (4. 6.) BA to AD, so is EA to AC; and consequently the rectangle BA, AC is equivalent (16. 6.) to the rectangle EA, AD. If, therefore, from an angle, &c. Q. E. D.

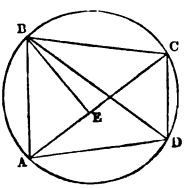


PROP. D. THEOR.

The rectangle contained by the diagonals of a quadrilateral inscribed in a circle, is equivalent to both the rectangles contained by its opposite sides.

Let ABCD be any quadrilateral inscribed in a circle, and join AC, BD; the rectangle contained by AC, BD is equivalent to the two rectangles contained by AB, CD, and by AD, BC.

Make the angle ABE equal to the angle DBC; add to each of these the common angle EBD, then the angle ABD is equal to the angle EBC. And the angle BDA is equal (21. 3.) to the angle BCE, because they are in the same segment: therefore the triangle ABD is equiangular to the triangle BCE: wherefore (4. 6.) as BC is to CE, so is BD to DA; and consequently the rectangle BC, AD is equivalent (16. 6.) to the rectangle BD, CE. Again, because the angle ABE is equal to the angle DBC, and the angle (21. 3.) BAE to the angle



BDC, the triangle ABE is equiangular to the triangle BCD: as, therefore, BA to AE, so is BD to DC; wherefore the rectangle BA, DC is equivalent to the rectangle BD, AE. But the rectangle BC, AD has been shown to be equivalent to the rectangle BD, CE; therefore the whole rectangle AC, BD (1. 2.) is equivalent to the rectangle AB, DC, together with the rectangle AD, BC. Therefore the rectangle, &c. Q. E. D.

ELEMENTS

OF

GEOMETRY.

ON THE CIRCUMFERENCES AND AREAS OF CIRCLES.

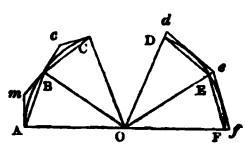
PRINCIPLE.

A straight line is the least of all those which can be drawn between two points.

And of two lines, one straight and the other curved, or each consisting of two or more straight or curve lines joining one another so as to form an angle or a series of angles, but all lying in the same plane, and terminating at the two extremities in the same points, having also their concavities towards the same side; that is the less which is contained within the other, except any part which may be common to both lines.

N. B. A curve line, or a line consisting of others which join so as to form angles, and of which all the parts lie in one plane, has its concavity towards the same side when no straight line can be drawn so as to cut the line in more than two points.

Hence, if the straight lines AB, BC form part of the periphery of a regular polygon inscribed in a circle, and A*m*, *m*B, B*c*, *c*C form part of the periphery of a regular polygon, of an equal number of sides, described about it, the straight lines and the circular arc ABC terminating in the same points A and C: the arc ABC is greater than the sum of the straight lines AB, BC, and less than the sum of the straight lines A*m*, *mc*, *c*C. Also the circumference of a



circle is greater than the perimeter of any polygon inscribed within it, and less than that of any polygon described about it.

Again. If DE, EF were sides of a regular polygon inscribed in the same circle, and if the sides of a regular polygon about the circle were drawn so as to touch the circle at points in the middles of the arcs DE, EF; then, since when the arc DEF is equal to the arc ABC and the polygons have the same number of sides, the sum of the lines *de*, *ef* is equal to the sum of the lines A*m*, *mc*, *c*C; it follows, O being the centre of the circle, that between any two radii, as OD, OF produced, the circular arc DEF is greater than the sum of the straight lines DE, EF, and less than the sum of the straight lines *de*, *ef*.

N. B. In order to abridge language, the following notations are sometimes adopted:

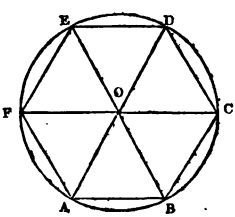
1. Circum. BA is used to denote the circumference of a circle described with the semidiameter BA;
2. Circle BA, to denote the area of a circle described with the semidiameter BA.

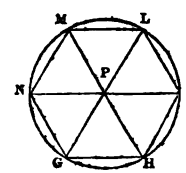
PROP. I. THEOR.

EQUILATERAL polygons, consisting of an equal number of sides, inscribed in circles, are equiangular to one another and similar; and are to one another in a duplicate ratio of their sides. Also the perimeters of the polygons are to one another in the same ratio as the sides.

Let ABCDEF, GHKLMN be two such equilateral polygons inscribed in the circles ABCD and GHKL: it is required to prove that they are equiangular to one another, &c.

Since each polygon is equilateral, it may be proved as in the eleventh proposition of the fourth book (Euclid) that it is equiangular; and since the two polygons have an equal number of sides, the sum of all the angles at the peripheries of the polygons are equal to one another (Euc. 1. Cor. 32. 1.): again, because the two polygons have an equal number of sides or angles,





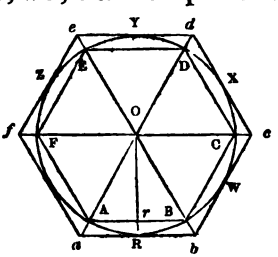
any angle, ABC, is the same part of the sum of all the angles at the periphery of one polygon as an angle, GHK, is of the sum of all the angles at the periphery of the other. But these sums are equal to one another; therefore each angle in one polygon is equal to each angle in the other, and the polygons are equiangular. And since they are also equilateral, the sides about the equal angles are proportional; therefore the polygons are similar (Euc. 1. Def. 6.): being similar, the polygons are to one another (Euc. 20. 6.) in a duplicate ratio of their sides, as AB and GH. Also (9. Pr.) the perimeters of the polygons are to one another in the same ratio as the sides.

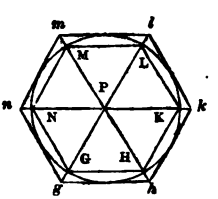
PROP. II. THEOR.

If the arcs cut off by the sides of a regular polygon inscribed in a circle be bisected, and at the points of bisection lines be drawn to touch the circle: the polygon thus formed will be equilateral and equiangular, and will be similar to the polygon inscribed in the circle.

Let ABCDEF be a regular polygon inscribed in the circle ABC as in the last proposition, and let the arcs AB, BC, CD, &c. be bisected in R, W, X, &c.; through R, W, X, &c., draw lines touching the circle in those points, and intersecting one another in *a*, *b*, *c*, &c.: the polygon *abcd*, &c. is equilateral and equiangular, and is similar to the polygon ABCD, &c.

Since the arcs AB, BC, CD, &c. are bisected in R, W, X, &c., the arcs RW, WX, &c. are equal to the arcs AB, BC, &c.; therefore, if the points R and W, W and X, &c. were joined by straight lines, there would be formed a polygon equal in every respect to the polygon ABCD, &c. It follows (Euc. 12. 4.) that the polygon *abcd*, &c. is equilateral and equiangular, and has the same number of sides as the polygon ABCD, &c.; consequently (1. Circle), the polygons are similar to one another. Q. E. D.





Cor. 1. Regular polygons *abcd*, &c., *ghkl*, &c., consist-

ing of an equal number of sides described about circles are similar figures (each of them being similar to a regular polygon (ABCD, &c., or GHKL, &c.) of an equal number of sides inscribed in either circle) and they are to one another in a duplicate ratio of their sides: also their perimeters are to one another in the same ratio as their sides. Again, if a regular polygon be inscribed in a circle, and a like polygon of an equal number of sides be described about the circle; the polygons will be to one another in a duplicate ratio of their sides, and their perimeters will be to one another in the same ratio as the sides.

Cor. 2. Since the sides of two regular polygons, consisting of an equal number of sides, one being inscribed in, and the other described about a circle, subtend equal angles at the centre, it is evident that the semidiameters OA, OB, &c., if produced, will pass through the points *a*, *b*, &c., and the triangles AOB, *aOb* will be equiangular; therefore AB : *ab* :: OA : O*a*. Now, joining the points O and R, the line OR, which bisects the angle AOB, will be perpendicular (Euc. 4. 1.) to AB, and (Euc. 18. 3.) to *ab*; and the angle RO*a* being common to the triangles RO*a*, *r*OA, those triangles are equiangular: therefore (Euc. 4. 6.), OA : O*a* :: O*r* : OR. Hence, by equality of ratios, AB : *ab* :: O*r* : OR, and (Euc. 22. 6.) $AB^2 : ab^2 :: Or^2 : OR^2$. But the polygons ABCD, &c. *abcd*, &c. are to one another in a duplicate ratio of their sides AB and *ab*, and their perimeters are to one another in the same ratio as those sides: therefore, if a regular polygon be inscribed in a circle, and a like polygon of an equal number of sides be described about it, the interior polygon is to the exterior polygon in a duplicate ratio of a perpendicular let fall from the centre of the circle upon one of its sides to a semidiameter of the circle: and the perimeter of the interior polygon is to that of the exterior polygon in the same ratio as the perpendicular is to the semidiameter.

Cor. 3. Let ABCD, &c., GHKL, &c., *abcd*, &c., and *ghkl*, &c. be regular polygons of an equal number of sides inscribed within and described about circles as in the above figure: then, since the triangles AOB, GPH are similar to one another, AB : GH :: AO : GP; whence $AB^2 : GH^2 :: AO^2 : GP^2$. But (Euc. 20. 6.) the polygon ABCD : the polygon GHKL :: $AB^2 : GH^2$; therefore those polygons are to one another as AO^2 is to GP^2; that is, regular polygons of an equal number of sides inscribed in circles are to one another in a duplicate ratio of the semidiameters of the circles. And since the perimeters of the polygons are to one another as AB is to GH, they are to one another as the semidiameters of the circles.

Also, since the four polygons ABCD, GHKL, *abcd*, *ghkl* are similar to one another and stand upon the four lines AB, GH, *ab*, *gh*, which are proportional; those polygons (Euc. 22. 6.) are proportional. It follows, that regular polygons of an equal number of sides described about circles are to one another in a duplicate ratio of the semidiameters of the circles; and that their perimeters are to one another in the same ratio as the semidiameters.

PROP. III. THEOR.

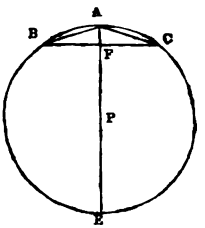
IF the circumference of a circle be bisected, and each half circumference, quarter circumference, and so on, again bisected (D. 4.), each arc so obtained will at length be less than any line that can be assigned. Then if lines be supposed to join every two nearest points of bisection, the figure so formed will be an equilateral and equiangular polygon inscribed in the circle.

The first part of this proposition depends on A. Pr.; each arc being less than any line that can be assigned, because it is that which remains when from the whole circumference there is taken its half, from the remainder its half, and so on continually. All the arcs are equal to one another, being constantly obtained by the bisection of equal arcs; therefore the lines joining the extremities of each arc are also equal to one another, and the polygon is equilateral: it may be proved to be equiangular as in the eleventh proposition of the fourth book of Euclid. Therefore, &c. Q. E. D.

PROP. IV. THEOR.

IF a regular polygon be inscribed in a circle, as in the last proposition, by successive bisections on the circumference till each arc is less than any assignable line, and then joining the points of division: the perpendicular let fall from the centre upon a side of the polygon will differ, in defect, from a semidiameter of the circle, by a line less than any that can be assigned.

Let BAC, obtained by continual bisections on the circumference, be a circular arc less than any assignable line, and



suppose it to be bisected in A. Draw the diameter AE, and suppose B and C, B and A, A and C to be joined: BC is the side of the equilateral and equiangular polygon, and it is bisected in F perpendicularly by the diameter AE (Euc. 30, 3. 3.). Now in the triangle ABF, the angle at F is greater than the angle at B; therefore BA is greater than AF (Euc. 19. 1.): but BA is less than the arc BAC, which is less than any assignable line; therefore AF, which is the difference between the perpendicular PF and the semidiameter PA, must be less than any assignable line. Q. E. D.

PROP. V. LEMMA.

IF by successive bisections on the circumference of a circle, and drawing lines touching the circle at the points of division, there be described a polygon consisting of more sides than four: the polygon will be regular, and its perimeter will be less than eight times the semidiameter of the circle.

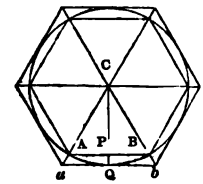
The regularity of the polygon may be inferred from the second proposition (Circle.) Now the side of a square being equal to a diameter of the circle, the perimeter of the square is equal to eight times the semidiameter: but the periphery of a polygon of 8, 16, 32, &c., sides being wholly within that of the square, except such parts as coincide with it, the perimeter is (Princip.) less than the perimeter of the square. Therefore, &c.

N. B. It may also be seen readily that any such polygon consisting of more sides than four, is less than four times as great as the square of the semidiameter of the circle.

PROP. VI. THEOR.

IF a regular polygon be inscribed in a circle, as in Prop. 3. (Circle.) by continual bisections on the circumference, till each arc is less than any assignable line, and if a regular polygon of an equal number of sides be described about the circle, as in Prop. 2. (Circle.): the perimeters of the polygons will differ from one another, and from the circumference of the circle, by lines less than any that can be assigned.

Let AB, *ab* be a side of each polygon, and from C, the centre of the circle, draw the semidiameter CQ to the point of contact, cutting AB in P. Then (2. Cor. 2. Circle) CQ : CP :: perimeter of polygon on *ab* : perim. of polygon on AB, and (6. Pr.) CQ : CQ − CP :: perim. of polygon on *ab* : difference between the perimeters of the polygons on *ab* and on AB; or (4. Pr.) CQ : perim. on *ab* :: CQ − CP : perim. on *ab* − perim. on AB. But (4. circle) CQ − CP is less than any line that can be assigned, and (5. circle) the perimeter of the polygon on *ab* (the second term of the proportion) is less than eight times as great as CQ (the first term); therefore (2. Cor. A. Prop.) perim. of polygon on *ab* − perim. of polygon on AB (the fourth term) is less than any line that can be assigned. But, again (Princip.), the circumference of the circle is greater than the perimeter of the inscribed polygon, and less than that of the other; therefore the circumference of the circle differs in excess and defect, respectively, from the perimeters of the polygons by lines less than any that can be assigned. Q. E. D.



PROP. VII. THEOR.

If a regular polygon be inscribed in a circle, and one be described about it as in the last proposition; the polygons shall differ from one another, and from the circle, by surfaces less than any that can be assigned. (Figure to the preceding proposition.)

For (2. cor. 2. circ.) $CQ^2 : CP^2$ :: polygon on *ab* : polygon on AB, and $CQ^2 : CQ^2 - CP^2$ :: polygon on *ab* : polygon on *ab* − polygon on AB; or alternately CQ^2 : polygon on *ab* :: $CQ^2 - CP^2$: polygon on *ab* − polygon on AB. But, since (4. circ.) CQ − CP is less than any line that can be assigned, $CQ^2 - CP^2$ is less (4. Cor. A. Pr.) than any surface that can be assigned; also the polygon on *ab* is less than four times as great as CQ^2; therefore (2. Cor. A. Pr.) polygon on *ab* − polygon on AB is less than any surface that can be assigned: but the circle is greater than the first polygon, and less than the second; therefore the circle differs from the two polygons, in defect and excess respectively, by surfaces less than any that can be assigned. Q. E. D.

N. B. It is evident that when two polygons differ, by surfaces less than any that can be assigned, from the circle

within which one of them is inscribed, and about which the other is described, their perimeters will differ from the circumference of the circle by lines less than any that can be assigned.

PROP. VIII. THEOR.

A CIRCLE is equivalent to a rectangle contained by its semidiameter, and a straight line equal to half its circumference.

Let a regular polygon (7, 6. Circ.) be inscribed in the circle, and a like polygon of an equal number of sides be described about it; the former differing from the circle in defect, and the latter in excess, by a surface less than any that can be assigned; also the perimeters of the polygons differing from the circumference of the circle, one in defect and the other in excess, by lines less than any that can be assigned.

Now the polygon inscribed within the circle is equal to the rectangle contained by a perpendicular let fall from the centre on one of its sides, and a line equal to the half perimeter (E. 4.); and the polygon described about the circle is equal to the rectangle contained by the semidiameter of the circle and a line equal to half the perimeter of the polygon. (the semidiameter being here the perpendicular let fall from the centre on one of the sides.) But the half perimeter of the first polygon differs from the half circumference of the circle, in defect, by a line less than any that can be assigned; and the half perimeter of the second polygon differs from the half circumference of the circle, in excess, by a line less than any that can be assigned: also the perpendicular in the first polygon differs from the semidiameter, in defect (4. Circ.) by a line less than any that can be assigned. Therefore (4. Cor. A. Pr.), the polygon inscribed in the circle will differ, in defect, from the rectangle of the semidiameter and a line equal to half the circumference by a surface less than any that can be assigned; and the polygon described about the circle will differ, in excess, from the rectangle of the semidiameter of the circle, and a line equal to its half circumference by a surface less than any that can be assigned.

It follows then, that both this rectangle (B) and the circle (A) differ, in excess and defect, from the polygons (A, and A′) inscribed in, and described about the circle, by surfaces less than any that can be assigned; therefore, by the First General

Theorem (Prop.), the circle, and the rectangle contained by the semidiameter and a line equal to the half circumference are equivalent to one another. Q. E. D.

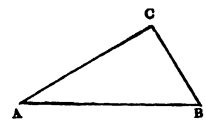
PROP. IX. THEOR.

CIRCLES are to one another in a duplicate ratio, or as the squares, of their diameters.

Let regular polygons (A, and B,) of an equal number of sides be inscribed in two circles (A and B), and like polygons (A′ and B′) of an equal number of sides be described about them; the inscribed polygons differing (7. Circ.) from the circles in defect, and the polygons circumscribed differing from them in excess, by surfaces less than any that can be assigned. All the polygons (1, 2. Circ.) are similar to one another; and the two inscribed polygons, as well as the two which are described about the circles, are to one another (3. Cor. 2. Circ.) in a duplicate ratio of the diameters of the circles. Therefore, by the Third General Theorem (Prop.), the circles are to one another in a duplicate ratio, or as the squares, of their diameters. Q. E. D.

COR. The circle described upon the side of a right-angled triangle, opposite the right angle, is equal to the two circles upon the other sides.

For let ABC be the triangle, right-angled at C; then $AC^2 : AB^2$ :: circle AC : circle AB, and $BC^2 : AB^2$ :: circle BC : circle AB; therefore (8. Pr.)



$AC^2 + BC^2 : AB^2$ :: circ. AC + circ. BC : circ. AB.

But (Euc. 47. 1.) $AC^2 + BC^2 = AB^2$; wherefore (2. Cor. Def. Pr.) circ. AC + circ. BC = circ. AB.

PROP. X. THEOR.

THE circumferences of circles are to one another as their diameters.

Let regular polygons (A, and B,) of an equal number of sides be inscribed in two circles (A and B), and like polygons (A′ and B′) of an equal number of sides be described about them; the perimeters (6. Circ.) of the inscribed polygons differing from the circumferences of the circles in defect,

and those of the polygons described about the circles differing from the circumferences in excess, by lines less than any that can be assigned. All the polygons (1, 2. Circ.) are similar to one another; and the perimeters of the inscribed polygons as well as those of the circumscribed polygons are (3. Cor. 2. Circles) to one another as the diameters of the circles. Therefore, by the Third General Theorem (Prop.), the circumferences of the circles are to one another as their diameters. Q. E. D.

ELEMENTS

OF

GEOMETRY.

GEOMETRY ON THE GROUND.

LINES, angles, and many geometrical figures may be traced on the ground by means of pickets, cords, and measuring tapes with tolerable accuracy; and the performance of such operations is frequently required for military purposes: among these may be mentioned, the encampment of troops, the determination of inaccessible distances preparatory to forming military bridges, opening trenches, or establishing artillery, and the construction of batteries or field fortifications of the various kinds. On these occasions a line is indicated, if not of great extent, by a cord stretched on the ground between two pickets; otherwise, by camp-colours, or some more conspicuous objects placed in the given direction. It may happen, moreover, that a circular arc is to be traced on the ground; in this case, a picket being driven at the place of the centre, one end of a cord of a proper length is made fast to it, and a man holding the other end keeps the cord tight. With this cord, as a radius, he describes the arc, which is usually distinguished by pickets driven into the ground at short intervals from one another.

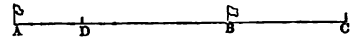
PROB. I.

To prolong a line which is already indicated by two pickets planted in the ground.

Let A and B be the two pickets; and let it be required to prolong the line towards the right or left hand.

The person who is to perform the operation must proceed to the right hand, if the line is to be produced in that di-

rection; and at any convenient distance he may plant a picket, as C, in such a position that it may seem to cover the objects at A and B. For the sake of greater accuracy he should stand a few feet beyond C in the direction of the given line produced, and he should observe that the sides of the three pickets are coincident.



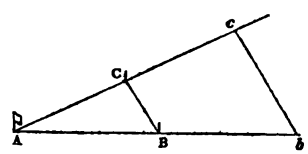
It may happen that a man, without any assistant, may be required to place a picket in the line between two given pickets, as between A and B: in this case, a picket must be first planted at C, as just described; and then the person may set up another, as at D, where it shall appear to cover the pickets B and C. That picket will be in the line between A and B.

PROB. II.

To trace on the ground a triangle the lengths of whose sides are given in feet or yards.

N. B. The tape measures, which are usually employed for these purposes, are 4 poles, or 66 feet, long: they are generally divided on one side, into feet and inches, and on the other, into links, each of which is equal to 7.82 inches. If none of the given sides should exceed 66 feet, the triangle may be easily traced by the help of two such tapes.

Let two pickets A and B, be fixed in the ground at a distance from one another equal to one of the given sides of the triangle as a base; that distance being measured by one of the tapes: then, a man holding together the places, on the two tapes, which are at distances from the ends equal to the two given sides of the triangle, two other men keep those ends to the ground at A and B: the first man then stretches the tapes, and when they are tight he drives a picket at the place where he stands, suppose at C: ABC is the triangle required.



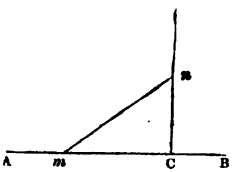
Should the lengths of the sides of the triangle be greater than the lengths of the tapes, any aliquot part, as one half, one third, &c., of the lengths of the sides must be taken, and the triangle formed with them as before: suppose ABC to be the triangle thus formed. Then, from any one angular point, two sides, as AB and AC must, by Prob. 1., be produced till the lines are respectively equal to the given lengths of the

sides. Suppose *b* and *c* to be the extremities of the sides so produced; then A*bc*, which is similar to ABC (Euc. 6. 6.), will be the required triangle.

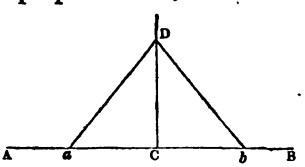
PROB. III.

To trace on the ground a line perpendicularly to a given line, from a given point in that line.

Let AB be the given line, and C the given point in it. If the required perpendicular is to be but a few feet in length, a triangle may be formed thus. Fold a portion of a tape, near one extremity, in three parts whose lengths are 3 feet, 4 feet, and 5 feet, and from C, lay on the ground towards A or B the part whose length is 4 feet: suppose this to be C*m*, and let the points C and *m*, on the tapes, be kept to the ground by two persons; then the parts C*n* and *nm* being 3 feet and 5 feet respectively, let a third person at *n*, stretch those parts of the tape tight, and the angle *m*C*n* will be a right angle (Euc. 47. 1.). For by the construction, the square of the hypotenuse *mn* is equal to the sum of the squares of the two sides C*m* and C*n*. If the required perpendicular is to be of considerable length, any convenient equimultiples of the above numbers may be taken on the tapes, and the triangle may be formed with them as before.



A simple process for raising a perpendicular, consists in measuring with a tape or string any equal distances C*a*, C*b* on the line AB; then having folded a string whose whole length is greater than *ab*, in two equal parts, let the two ends be held to the ground at *a* and *b*, and let a person holding the middle point D stretch the string. The line from D to C will be perpendicular to AB (Euc. 11. 1.).



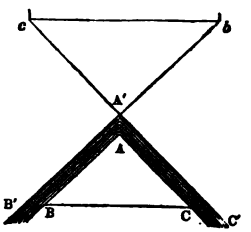
If the given point be on either side of the given line, and it were required to let fall a perpendicular from that point, upon the line; proceed thus. Let AB be the line, and D the point. Fold a cord in two equal parts, and while one person holds the middle point at D, let two persons stretch the parts so that the ends may be in the direction of AB, as at *a* and *b*. Then, if the distance between *a* and *b* be bisected,

suppose at C, DC will be the perpendicular required. If the points A and B are not connected by a cord on the ground, a person standing beyond A or B must observe that the ends *a* and *b* of the folded cord are kept in the direction of the given line.

PROB. IV.

To measure on the ground the angle contained between any two accessible lines.

First. When the angle, as BAC, is situated between the interior slopes or walls of a rampart at what is called a salient point, or between the exterior slopes at what is called a re-entering point. Drive a picket into the ground at A, the angular point: then, with a tape, measure any equal distances AB, AC along the sides containing the angle; and again, with the tape, measure the distance from B to C. The knowledge of the three sides AB, AC, BC, determines the angle BAC.

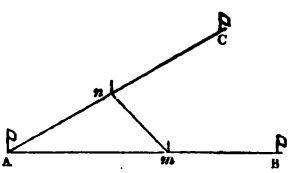


Secondly. When the angle, as B′A′C′, is between the exterior sides of a rampart at a salient point. With a tape, measure any equal distances, A′*b*, A′*c*, in the directions of the sides B′A′, C′A′ produced; and again, measure the distance from *b* to *c*. The knowledge of the sides A′*b*, A′*c*, *cb* determines the angle *c*A′*b*, or its equal (Euc. 15. 1.) the angle B′A′C′.

PROB. V.

To trace on the ground an angle equal to a given angle; the angular point and the direction of one of the sides, on the ground, being given.

Whether the given angle be obtained by admeasurement on the ground as in the last problem, or whether it be described on paper, the lengths of three lines, between two of which the angle is situated, are supposed to be known: therefore, if AB be the given line, and A the angular point, measure from A towards B a distance A*m* equal to one of the

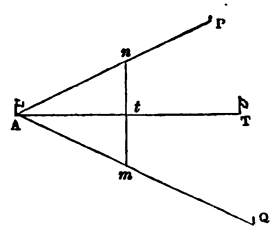


sides containing the angle, and upon Am construct with tapes, by Prob. 2. a triangle, as Anm, of which the sides An, mn are equal to the given lengths of the corresponding sides. Then mAn on the ground will be equal to the given angle; and, by pickets set up at A and n, the line An may be produced to any required extent.

PROB. VI.

To trace on the ground, through a given point, a line bisecting the angle contained between the directions in which two visible objects lie from that point.

Let A be the angular point, P and Q the two objects. Set out, with a tape, any equal distances An, Am in the directions of the objects; and again with a tape, measure the distance nm in feet or otherwise. Plant a picket at t in the middle of the line nm, and prolong At to any required distance, as AT, from A. The line AT bisects the angle PAQ (Euc. 8. 1.).

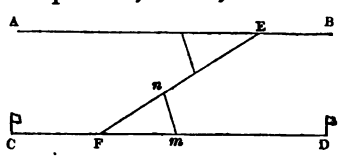


PROB. VII.

THROUGH any given point on the ground, to trace a line parallel to a given accessible line.

Let F be the given point, and AB the given line.

Place a picket at any convenient point E, in AB, and with tapes, by Prob. 4., measure the angle AEF. Then at F, in the line supposed to join F and E, lay down by Prob. 5., an angle as mFn equal to AEF. The line Fm produced in opposite directions, if necessary, will be parallel to AB. (Euc. 27. 1.)

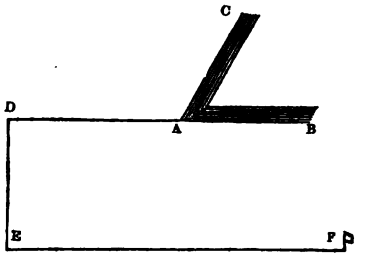


PROB. VIII.

To trace on the ground a line parallel to some line, and at a given distance from it; the prolongation only of the given line being accessible.

Let AB be the line parallel to which it is required to trace one on the ground at some given distance.

Set up a picket at any convenient point D in the prolongation of BA; at D trace by Prob. 3. a line perpendicular to DA, towards the ground on which the required line is to be traced. On the direction of this perpendicular measure, with tapes or otherwise, the given distance; and let the length of DE be equal to that distance. At E trace, by the same problem, a perpendicular to DE in the required direction; and let EF be this perpendicular. The line EF will be parallel to AB (Euc. 28. 1.).

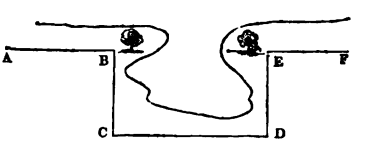


N. B. This problem may be put in practice when it is required to place a battery opposite, and parallel to a given line, as AB.

PROB. IX.

To prolong the direction of a line through an obstacle, even though the ground beyond the obstacle should be concealed from the view of the spectator in any part of the given line.

Let AB be the given line: at B trace perpendicularly to AB a line, as BC, of any convenient length, so that it may extend beyond the head of the obstacle. At C trace perpendicularly to BC, consequently parallel to AB (Euc. 27. 1.), a line, as CD, of any convenient length. Again, at D trace perpendicularly to CD a line, as DE, which make equal in length to BC: lastly, trace perpendicularly to DE a line, as EF. This line will be in the prolongation of AB.



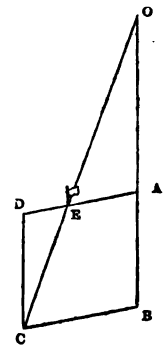
Since BC is equal and parallel to ED, BE is equal and parallel to CD (Euc. 33. 1.), and it is also coincident in direction with AB and EF; therefore the lengths of AB, CD, and EF being measured, their sum will be equal to the distance of A from F.

N. B. This problem is occasionally put in practice in the processes of mining.

PROB. X.

To find the distance of an inaccessible object from any given point.

Let O be the inaccessible object, and A or B the given point. In a direction from or towards the object, according as A or B is the given point, measure any convenient distance, AB, with a tape or cord; and in any convenient direction, measure a distance, as BC, equal to AB. With two tapes or cords, each equal to AB or BC, construct a rhombus, as DB, by causing two men to keep on the ground an end of one tape or cord at A, and an end of the other at C; or by making those ends fast to pickets at the same points: then, holding in the hand the other ends of both, stretch the tapes or cords; these will take the directions AD and CD: lastly, place a picket at E on the line AD, and in the direction of the visual ray OC. The triangles EDC and EAO, or CBO, are equiangular and similar; for by the nature of the rhombus or parallelogram, the angle at D is equal to CBA (Euc. 34. 1.) or DAO; and the angle DEC is equal to OEA or OCB (Euc. 15, 29. 1.); therefore the angle DCE is equal to the angle COA: consequently (Euc. 4. 6.)

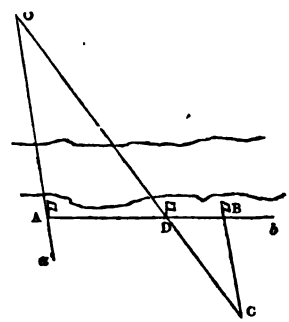


ED : DC :: AE : AO,
or ED : DC :: CB : BO.

Hence, since the three first terms can be obtained by measurement on the ground, the distance AO or BO may be computed.

Otherwise,

Let O be the inaccessible object and A the given point; set up pickets at A, and at any convenient points, as D and B, in the direction AB, which may be taken at pleasure. By Prob. 4, measure the angle OAD, or the adjacent angle *a*AD, as may be most convenient: also measure the distances AD and DB. At B, by Prob. 5, lay down the angle DBC equal to OAD, or *b*BC equal to *a*AD; then BC will be parallel to OA, and if the point C be taken in BC, in the direction of the visual ray OD, the triangle DBC will be equiangular and similar to OAD. Therefore



DB : BC :: AD : AO;

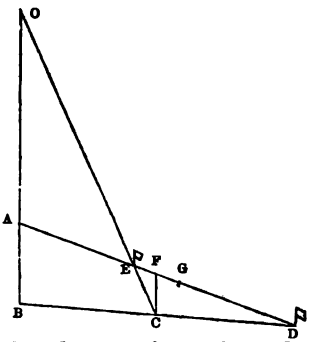
and BC, as well as AD and DB being measured, the distance AO may be computed.

Again,

Let O be the inaccessible object, and A or B be the given point. (these two points being supposed to be in the same direction from O.)

Set up pickets at A and B, which may be at any convenient distance from one another; and, in any convenient direction from B, measure equal distances BC and CD: also set up a picket at E in the directions of the visual rays OC and AD. Then

ED—AE : AB :: AE : AO.



For, imagine CF to be drawn parallel to AB; then because BD is bisected in C, AD will be bisected in F (Euc. 2. 6.). The triangle EFC is equiangular, and similar to EAO; for the angles at E are equal to one another, being vertically opposite, the angle EFC is equal to OAE, and the angle ECF is equal to AOE, they being alternate angles: therefore (Euc. 4. 6.)

EF : FC :: AE : AO,

or (1. Pr.) 2EF : 2FC :: AE : AO.

But supposing DG to be made equal to AE, EG will be the difference between AE and ED; and since AF=FD, EF will be equal to FG (Euc. 3. Ax. 1.): therefore 2EF=EG or 2EF =ED—AE. Again, since DB=2DC, AB=2FC (Euc. 4. 6.); therefore, by substitution, the last proportion becomes

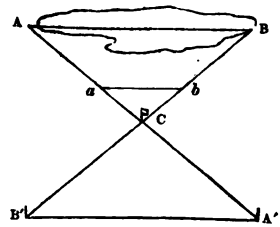
ED—AE : AB :: AE : AO;

Therefore, AB, AE, and ED being measured on the ground, the length of AO may be computed.

PROB. XI.

To find the length of a line which is accessible at both its extremities.

Let AB be the given line, which is accessible at A and B. At any convenient point as C, set up a picket, and measure the distances AC and BC: then, if the ground will

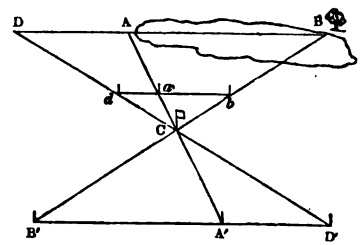


permit it, prolong these lines till CA′ is equal to CA, and CB′ to CB, and measure A′B′. This last line will be equal to the required distance AB (Euc. 4. 1.). Should the ground not permit the prolongation of AC and BC; take one half, one third, or any other aliquot part of each of those lines, and measure distances, as C*a* and C*b*, equal to the lengths of the parts. The line *ab* will be parallel to AB (Euc. 2. 6.), and AB will be the same multiple of *ab*, as CA or CB is of C*a* or C*b* (Euc. 6. 6.). Therefore AB may be found.

PROB. XII.

To find the length of a line which is accessible at one extremity only.

Let AB be the line, which is terminated by a conspicuous object at B, but is accessible at A only.



At any convenient point as C, set up a picket; also set up a picket at any point D, in the prolongation of the visual ray, BA, and measure AC, DC. If the ground will permit it, prolong these lines till CA′ is equal to CA, and CD′ is equal to CD. Plant pickets at A′ and D′, and carry a picket in the direction D′A′ till the prolongation of this line meets that of the visual ray BC: let the point of meeting be B′. Then, since CA=CA′, CD=CD′, and the vertically opposite angles ACD, A′CD′ are equal, the triangle CAD and A′CD′ are equal, and the angle CAD is equal to CA′D (Euc. 4. 1.); therefore the angle CAB is equal to CA′B′. Now, in the triangles CAB and CA′B′, the angle CAB is equal to CA′B′, CA is equal to CA′ by construction, and the angle ACB is equal to the vertically opposite angle A′CB′: therefore (Euc. 26. 1.) AB=A′B′. Consequently, on measuring the length of A′B′, that of AB is found.

Should the ground not permit the lines AC, DC to be prolonged, measure C*a*, C*d* equal to any like aliquot part of the lengths of the lines CA, CD, and set up a picket at *b*, where

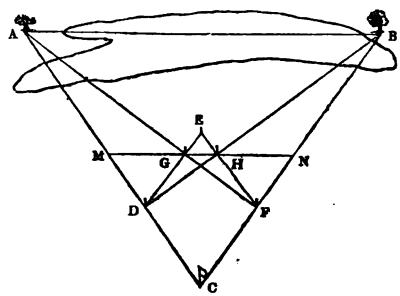
the prolongation of the line *da* meets the visual ray BC: then, since *dab* is parallel to DAB, as in the last problem, AB will be the same multiple of *ab* as CA is of C*a*. Therefore *ab* being measured, the length of AB may be found.

PROB. XIII.

To find the length of a line which is entirely inaccessible.

Let AB be the given line, of which the two extremities are visible, though inaccessible.

Set up a picket at any convenient point C, and in the directions CA, CB measure any equal distances CD, CF with tapes. On these lines, with tapes or cords, form the rhombus DF, as in Prob. 10., and set up pickets at G and H in the intersections of the visual rays AF, DE and DB, EF.



Since, as in Prob. 10., the triangle GEF is similar to FCA, and the triangle HED to DCB,

GE : EF :: FC : CA, and HE : ED :: DC : CB (Euc. 4. 6.); therefore GE, CA = EF, FC and HE, CB = ED, DC (Euc. 16. 6.). But the rectangle EF, FC is equal to ED, DC, since all the sides of the rhombus are equal to one another. Therefore

GE, CA = HE, CB, and (Euc. 16. 6.) GE : HE :: CB : CA.

Hence, in the triangles GEH and ACB, the sides are proportional about the equal angles at E and C; therefore those triangles are similar (Euc. 6. 6.), and

EH : HG :: CA : AB.

The lines HE, EG and HG can be measured on the ground, and the sides of the rhombus are known: therefore, CA being computed from the similar triangles GEF, ACF, and its value substituted in the last proportion, the length of AB may be found.

Let the line of the pickets at G and H be produced to meet the visual rays from A to C and from B to C, in M and N; then since, from the similarity of the triangles GEH and ABC, the angle EHG is equal to CAB, the line GH or MN is parallel

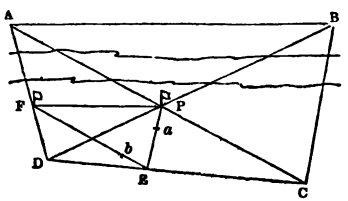
to AB (Euc. 28. 1.). Thus a line has been traced on the ground, parallel to an inaccessible line: and we have, moreover, CM : MN :: CA : AB.

It should be observed, however, that the position of MN, and the computed length of AB, will be very liable to inaccuracy, on account of the smallness of the triangle GEH, upon which both depend.

PROB. XIV.

THROUGH a given point to trace on the ground a line parallel to an inaccessible line.

Let AB be the inaccessible line, and P the given point. Set up a picket at any convenient point C in the direction of a visual ray passing through A and P, and another at a convenient point D in the direction BP. At C with tapes, by Prob. 4., measure the angle ACB; and at P, on PC lay down by Prob. 5., an equal angle CP*a*. Leaving pickets at P and *a*, place another picket at E in the direction of P*a* prolonged, and also in the direction of the visual ray from C to D: then the line PE will be parallel to BC (Euc. 27. 1.) Again, at E, on EP, make by Prob. 5., an angle PE*b* equal to ACB or CPE; and leaving pickets at E and *b*, place one at F in the direction of E*b* prolonged, and of the visual ray from A to D: then the line EF will be parallel to AC, and the line of the pickets P and F will be parallel to AB.



For because PE is parallel to BC,

DE : DC :: DP : DB (Euc. 2. 6.)

Again, because FE is parallel to AC,

DE : DC :: DF : DA.

Therefore, by equality of ratios, DP : DB :: DF : DA; and consequently (Euc. 2. 6.), FP is parallel to AB.

If it be required to find the length of AB, the following process may be used. In the similar triangles FDE, ADC;

DE : DC :: DF : DA (Euc. 2. 6.);

And again, in the similar triangles DFP, DAB;

DF : DA :: FP : AB (Euc. 4. 6.):

Therefore, by equality of ratios,

DE : DC :: FP : AB.

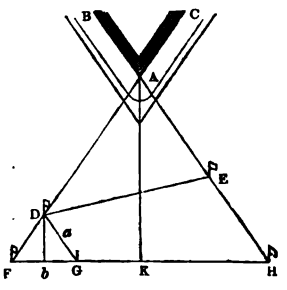
Hence, if DE, DC and FP be measured on the ground, the length of AB may be found by computation.

PROB. XV.

To bisect an inaccessible salient angle, or to find on the ground the prolongation of the capital of an inaccessible work.

Let BAC be the inaccessible angle, and let it be presumed that the directions of the sides BA and CA may be obtained on the ground.

Set up pickets at D and E, any convenient points in the prolongations of CA and BA: at E, by Prob. 4., measure the angle AED, and at D, by Prob. 5., make the angle EDa equal to AED: the line Da will be parallel to BAE. Prolong that line, if necessary, to any convenient length DG, and place a picket at G. Measure DG; make DF, in the prolongation of AD, equal to DG, and place a picket at F. Set up a picket at the intersection of the visual rays through FG and BA: let this intersection be at H, which, it may be observed, may fall either between A and E, or, as in the figure, beyond E with respect to A; and having measured the length of FH, find its middle point K. A line from A through K will bisect the angle DAE, or the opposite angle BAC; that is, it will be the prolonged capital of the work. For the triangle FDG is similar to FAH (Euc. 29. 1., 4. 6.) since DG is parallel to AH by construction, and the angle at F is common: but DF is equal to DG; therefore AF is equal to AH. Again, FK is equal to KH by construction, AF is equal to AH, and AK is common; therefore the angle FAK is equal to KAH (Euc. 8. 1.): that is, the angle FAH or BAC is bisected by the line AK.



The length of AK may be found on bisecting FG in b and measuring Fb and Db; for, on account of the similarity of the triangles DFb and AFK,

$$F b : b D :: FK : AK;$$

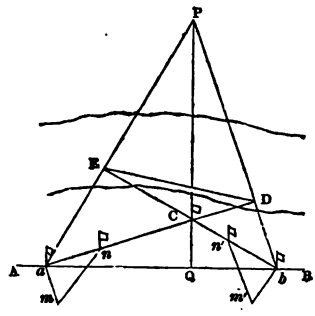
of which proportion the three first terms are known.

PROB. XVI.

In a given line on the ground to find the point at which a perpendicular supposed to be let fall from an inaccessible object will meet the line.

Let AB be the given line, and P the inaccessible point: it is required to find the point Q, at which a line from P, perpendicular to AB, will meet that line.

Take any two points a and b in the line, in such situations that the angles PaB, PbA may be less than right angles, and with tapes, by Prob. 4., measure the angles PaB, PbA. At a form, by Prob. 5., an angle Bam equal to PbA: then will am be parallel to Pb. On am construct, by Prob. 3., a right angle, as man, and set up pickets at a and n: then will an be perpendicular to am, or Pb. In like manner at b make the angle Abm' equal to the measured angle PaB; construct the right angle $m'bn'$, and set up pickets at b and n': then will bn' be perpendicular to bm', or Pa. Find a point C at the intersection of the visual rays in the directions an and bn', and there place a picket. Lastly, in the line AB set up a picket at Q in the direction of the visual ray passing through P and C: then will PQ be perpendicular to AB. For, imagine aC and bC to be produced to D and E, and join D, E: because the angles CDP and CEP are right angles, a circle will pass through the points C, E, P, D, (Euc. 22. 3.); and the angle CED, or bED will be equal to CPD, or bPQ (Euc. 21. 3.). Again, because each of the angles aEb and aDb is a right angle, a circle will pass through a, E, D, b (Euc. 31. 3.); and the angle baD will be equal to bED; therefore the angle baD is equal to QPb; also the angle PbQ is common to the two triangles baD and QPb: therefore the angle aDb is equal to PQb; but the angle aDb is a right angle: therefore the angle PQb is a right angle, and PQ is perpendicular to AB.



The triangle aCQ is similar to the triangle PQb; for the angle QaC is equal to QPb, and the adjacent angles at Q are right angles: therefore (Euc. 4. 6.)

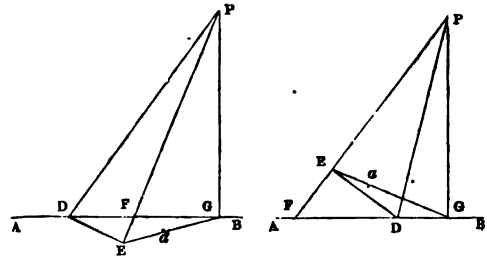
$$\text{CQ} : a\text{Q} :: \text{Q}b : \text{QP}.$$

Consequently, the three first lines being obtained by admeasurement, the length of QP may be found by computation.

Otherwise, let AB be the given line, and P the inaccessible point.

Set up a picket at any convenient point E, on either side of the given line, and at that point lay down a right angle, PED, having a visual ray supposed to join P and E for one of

its sides: place a picket at D, in the line AB, and in the direction of the other side; and measure the angle PDB with tapes or otherwise. Again, at E lay down an angle PE*a* equal to PDB, having a ray supposed to join P and E for one of the sides containing the angle: then, if a picket be set up at G, in the line AB, and in the direction of the other side E*a*, the line PG will be perpendicular to AB.

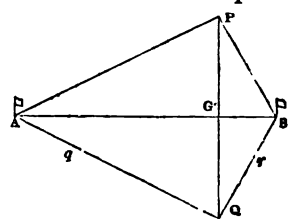


For, since the angle PEG is equal to PDG by construction, those angles are in the same segment of a circle, and that circle will pass through the points P, E, D, G, (Euc. 21. 3.); therefore the angle DEP is equal to PGD: but DEP is a right angle by construction; therefore PGD is a right angle.

Let F be the intersection of PE, produced, if necessary, with AB: then the triangle DEF will be equiangular and similar to PGF; and (Euc. 4. 6.)

EF : ED : : FG : PG.

Again, let AB be the given line, and P the inaccessible point. At A, measure the angle PAB, and on the opposite side of AB lay down BA*q* equal to that angle: also at B, measure the angle PBA, and on the opposite side of AB lay down AB*r* equal to that angle. Leaving pickets at A and *q*, and at B and *r*, find the point Q in the intersection of the two lines of pickets; then the line PQ will be perpendicular to AB (Euc. 8. 1.).



Let the intersection G, of AB and PQ be found; then, from the equality of the triangles PAG, QAG (Euc. 4. 1.), it is evident that the inaccessible distance PG will be equal to GQ.

N.B. It may be sometimes convenient to put the last method in practice in addition to the first of the constructions given in this problem. Thus, after the directions of the lines *an* and *bn'* have been found, upon the line AB and on the opposite side of it let angles equal to *nab* and *n'ba* be made at the points *a* and *b* respectively; then it is evident that the two lines laid down below AB will, if produced, intersect one another in a point which will be in the direction of a line passing through P perpendicularly to AB.

If on the ground there be given a line on which, as a side, it is required to construct an equilateral and equiangular figure of three, four, five, six, or eight sides; constructions derived from Euc. 1. 1., Euc. 46. 1., and Euc. A., B., C., 4., may be employed.

ELEMENTS

OF

GEOMETRY.

THE INTERSECTIONS OF PLANES.

DEFINITIONS.

I.

A STRAIGHT line is perpendicular, or at right angles to a plane, when it makes right angles with every straight line in that plane, which it meets.

II.

A plane is perpendicular to a plane, when the straight lines drawn in one of the planes perpendicular to the common section of the two planes, are perpendicular to the other plane.

Hence, if a line be drawn in one of two perpendicular planes, at right angles to their common section, it will be perpendicular to the other plane.

III.

The inclination of a straight line to a plane, is the acute angle contained by that straight line, and another drawn from the point in which the first line meets the plane, to the point in which a perpendicular to the plane, drawn from any point in the first line, meets the same plane.

IV.

The angle made by two planes which cut one another, is the angle contained by two straight lines drawn from any one point in the line of their common section, at right angles to that line, the one in the one plane, and the other

in the other. Of the two adjacent angles made by two lines drawn in this manner, that which is acute is also called the inclination of the planes to one another.

V.

Two planes are said to have the same, or a like inclination to one another, which two other planes have, when the angles of inclination above defined are equal to one another.

VI.

A straight line is said to be parallel to a plane, when it does not meet the plane, though produced ever so far.

VII.

Planes are said to be parallel to one another, which do not meet, though produced ever so far.

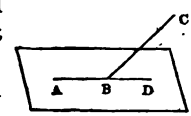
VIII.

A solid angle is an angle made by the meeting of more than two plane angles, which are not in the same plane, in one point.

PROP. I. THEOR.

ONE part of a straight line cannot be in a plane and another part above it.

If it be possible, let AB, part of the straight line ABC, be in the plane, and the part BC above it: and since the straight line AB is in the plane, it can be produced in that plane (Euc. 2. Post. 1.): let it be produced to D. Then ABC and ABD are two straight lines, and they have the common segment AB, which is impossible (Euc. Cor. 11. 1.). Therefore ABC is not a straight line. Wherefore, one part, &c. Q. E. D.



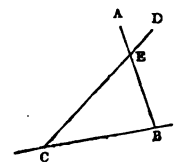
PROP. II. THEOR.

ANY three straight lines which meet one another, not in the same point, are in one plane.

Let the three straight lines AB, CD, CB meet one another in the points B, C and E: AB, CD, CB, are in one plane.

Let any plane pass through the straight line EB, and let

the plane be turned about EB, produced, if necessary, until it pass through the point C: then, because the points E, C are in this plane the straight line EC is in it (Euc. 7. Def. 1.): for the same reason, the straight line BC is in the same; and by the hypothesis, EB is in it: therefore the three straight lines EC, CB, BE are in one plane: but the whole of the lines DC, AB, and BC, produced, are in the same plane with the parts of them EC, EB, BC (1. Planes.). Therefore, AB, CD, CB are all in one plane. Wherefore, &c. Q. E D.

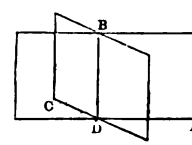


COR. It is manifest, that any two straight lines which cut one another are in one plane: also, that any three points whatever are in one plane.

PROP. III. THEOR.

IF two planes cut one another: their common section is a straight line.

Let two planes AB, BC cut one another, and let B and D be two points in the line of their common section. From B to D draw the straight line BD; and because the points B and D are in the plane AB, the straight line BD is in that plane (Euc. 7. Def. 1.); for the same reason it is in the plane CB; the straight line BD is therefore common to the planes AB and BC, or it is the common section of these planes. Therefore, &c. Q. E. D.



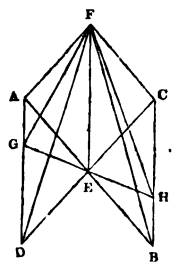
PROP. IV. THEOR.

IF a straight line stand at right angles to each of two straight lines in the point of their intersection: it will also be at right angles to the plane in which these lines are.

Let the straight line EF stand at right angles to each of the straight lines AB, CD in E, the point of their intersection: EF is also at right angles to the plane passing through AB, CD.

Take the straight lines AE, EB, CE, ED all equal to one another: and through E draw, in the plane in which are AB, CD, any straight line GEH; and draw AD, CB; then, from

any point F in EF, draw FA, FG, FD, FC, FH, FB. And, because the two straight lines AE, ED are equal to the two BE, EC, and that they contain equal angles (Euc. 15. 1.) AED, BEC, the base AD is equal (Euc. 4. 1.) to the base BC, and the angle DAE to the angle EBC: and the angle AEG is equal to the angle BEH (Euc. 15. 1.); therefore the triangles AEG, BEH have two angles of one equal to two angles of the other, each to each, and the sides AE, EB, adjacent to the equal angles, equal to one another; wherefore they shall have their other sides equal (Euc. 26. 1.): GE is therefore equal to EH, and AG to BH. And because AE is equal to EB, and FE common and at right angles to them, the base AF is equal (Euc. 4. 1.) to the base FB; for the same reason, CF is equal to FD: and because AD is equal to BC, and AF to FB, the two sides FA, AD are equal to the two FB, BC, each to each; and the base DF was proved equal to the base FC; therefore the angle FAD is equal (Euc. 8. 1.) to the angle FBC. Again,



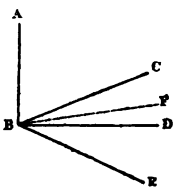
it was proved that GA is equal to BH, and also AF to FB; FA, then, and AG, are equal to FB and BH, and the angle FAG has been proved equal to the angle FBH: therefore the base GF is equal (Euc. 4. 1.) to the base FH. Again, because it was proved that GE is equal to EH, and EF is common; GE, EF are equal to HE, EF; and the base GF is equal to the base FH; therefore, the angle GEF is equal (Euc. 8. 1.) to the angle HEF; and consequently each of these angles is a right (Euc. 10. Def. 1.) angle. Therefore FE makes right angles with GH, that is, with any straight line drawn through E in the plane passing through AB, CD. In like manner it may be proved that FE makes right angles with every straight line which meets it in that plane. But a straight line is at right angles to a plane, when it makes right angles with every straight line which meets it in that plane (1. Def. Pl.): therefore EF is at right angles to the plane in which are AB, CD. Wherefore, if a straight line, &c. Q. E. D.

PROP. V. THEOR.

If three straight lines meet all in one point, and a straight line stand at right angles to each of them in that point: these three straight lines are in one and the same plane.

Let the straight line AB stand at right angles to each of the straight lines BC, BD, BE, in B, the point where they meet: BC, BD, BE are in one and the same plane.

If not, let, if it be possible, BD and BE be in one plane, and BC be above it; and let a plane pass through AB, BC, the common section of which with the plane in which BD and BE are, shall be a straight (3. Pl.) line; let this be BF: therefore the three straight lines AB, BC, BF are all in one plane, viz. that which passes through AB, BC: and because AB stands at right angles to each of the straight lines BD, BE, it is also at right angles (4. Pl.) to the plane passing through them; and therefore makes right angles with every straight line meeting it in that plane; but BF, which is in that plane, meets it: therefore the angle ABF is a right angle; but the angle ABC, by the hypothesis, is also a right angle; therefore the angle ABF is equal to the angle ABC, and they are both in the same plane, which is impossible; therefore the straight line BC is not above the plane in which are BD and BE: wherefore, the three straight lines BC, BD, BE are in one and the same plane. Therefore, if three straight lines, &c. Q. E. D.

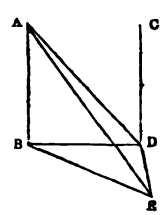


PROP. VI. THEOR.

IF two straight lines be at right angles to the same plane: they shall be parallel to one another.

Let the straight lines AB, CD be at right angles to the same plane BDE: AB is parallel to CD.

Let them meet the plane in the points B, D. Draw DE at right angles to DB, in the plane BDE, and let E be any point in it: join A, E, A, D, E, B. Because ABE is a right angle, $AB^2 + BE^2 =$ (Euc. 47. 1.) AE^2; and because BDE is a right angle, $BE^2 = BD^2 + DE^2$; therefore $AB^2 + BD^2 + DE^2 = AE^2$; now, $AB^2 + BD^2 = AD^2$, because ABD is a right angle; therefore $AD^2 + DE^2 = AE^2$, and ADE is therefore a (Euc. 48. 1.) right angle: therefore ED is perpendicular to the three lines BD, DA, DC, whence these lines are in one plane (5. Pl.). But AB is in the plane in which are BD, DA, because any three straight lines, which meet one another, are in one plane (2. Pl.); therefore AB, BD, DC are in one plane; and each



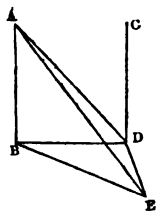
of the angles ABD, BDC is a right angle; therefore AB is parallel (Euc. 29. 1.) to CD. Wherefore, if two straight lines, &c. Q. E. D.

PROP. VII. THEOR.

If two straight lines be parallel, and one of them at right angles to a plane: the other is also at right angles to the same plane.

Let AB, CD be two parallel straight lines, and let one of them, AB be perpendicular to a plane: the other, CD is at right angles to the same plane.

Let AB and CD meet the plane in the points B and D: draw DE perpendicular to BD in the same plane, and let E be any point in it: join A,D, A,E, B,E. Because AB is perpendicular to the plane BDE, the angles ABD and ABE are right angles; wherefore $AE^2 = AB^2 + BE^2$ (Euc. 47. 1.); and because BDE is a right angle, $BE^2 = BD^2 + DE^2$ (Euc. 47. 1.); consequently $AE^2 = AB^2 + BD^2 + DE^2$: now $AD^2 = AB^2 + BD^2$ (Euc. 47. 1.); therefore $AE^2 = AD^2 + DE^2$. Wherefore ED is perpendicular to AD (Euc. 48. 1.); but ED is also perpendicular to DB; therefore it is perpendicular to the plane of the parallels AB and DC, in which the lines AD and DB are situated (4. Pl.): the line DE is therefore perpendicular to DC. And because DC is parallel to AB, and AB is perpendicular to DB, DC is also perpendicular to DB (Euc. 29. 1.): wherefore DC is perpendicular to the plane in which the lines DB and DE are contained (4. Pl.), that is, it is perpendicular to the same plane to which AB is perpendicular. Therefore, if two straight lines, &c. Q. E. D.



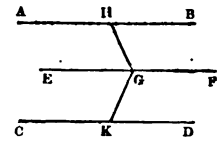
PROP. VIII. THEOR.

Two straight lines, which are each of them parallel to the same straight line, though not both in the same plane with it, are parallel to one another.

Let AB, CD be each of them parallel to EF, and not in the same plane with it: AB shall be parallel to CD.

In EF take any point G, from which draw, in the plane passing through EF, AB, the straight line GH at right angles to EF; and in the plane passing through EF, CD, draw GK

at right angles to the same EF. And because EF is perpendicular both to GH and GK, EF is perpendicular (4. Pl.) to the plane HGK passing through them: and EF is parallel to AB; therefore AB is at right angles (7. Pl.) to the plane HGK. For the same reason, CD is likewise at right angles to the plane HGK: therefore AB, CD are each of them at right angles to the plane HGK. But if two straight lines are at right angles to the same plane, they are parallel (6. Pl.) to one another: therefore AB is parallel to CD. Wherefore two straight lines, &c. Q. E. D.

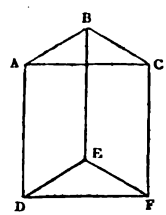


PROP. IX. THEOR.

IF two straight lines meeting one another be parallel to two others that meet one another, though not in the same plane with the other two: the first two and the other two shall contain equal angles.

Let the two straight lines AB, BC, which meet one another, be parallel to the two straight lines DE, EF, that meet one another, and are not in the same plane with AB, BC: the angle ABC is equal to the angle DEF.

Take BA, BC, ED, EF all equal to one another, and draw AD, CF, BE, AC, DF. Because BA is equal and parallel to ED, therefore AD is (Euc. 33. 1.) both equal and parallel to BE. For the same reason, CF is equal and parallel to BE: therefore, AD and CF are each of them equal and parallel to BE. But straight lines that are parallel to the same straight line, though not in the same plane with it, are parallel (8. Pl.) to one another: therefore AD is parallel to CF; and it is equal to it, and AC, DF join them towards the same parts; and therefore (Euc. 33. 1.) AC is equal and parallel to DF. And because AB, BC are equal to DE, EF, and the base AC to the base DF; the angle ABC is equal (Euc. 8. 1.) to the angle DEF. Therefore, if two straight lines, &c. Q. E. D.

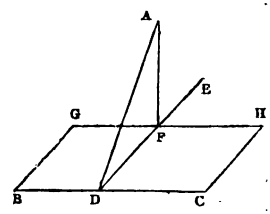


PROP. X. PROB.

To draw a straight line perpendicular to a plane, from a given point above it.

Let A be the given point above the plane BH: it is required to draw from the point A a straight line perpendicular to the plane BH.

In the plane draw any straight line BC, and from the point A draw (Euc. 12. 1.) AD perpendicular to BC. If then AD be also perpendicular to the plane BH, the thing required is already done: but if it be not, from the point D draw (Euc. 11. 1.) in the plane BH, the straight line DE, at right angles to BC; and from the point A draw AF perpendicular to DE; and through F draw (Euc. 31. 1.) GH parallel to BC; and because BC is at right angles to ED and DA, BC is at right angles (4. Pl.) to the plane passing through ED, DA. And GH is parallel to BC; but if two straight lines be parallel, one of which is at right angles to a plane, the other shall be at right (7. Pl.) angles to the same plane; wherefore GH is at right angles to the plane passing through ED, DA, and is perpendicular (1 Def. Pl.) to every straight line meeting it in that plane. But AF, which is in the plane through ED, DA, meets it: therefore GH is perpendicular to AF; and consequently AF is perpendicular to GH: and AF is also perpendicular to DE: therefore AF is perpendicular to each of the straight lines GH, DE. But if a straight line stands at right angles to each of two straight lines in the point of their intersection, it shall also be at right angles to the plane passing through them (4. Pl.). And the plane passing through ED, GH is the plane BH; therefore AF is perpendicular to the plane BH; so that, from the given point A, above the plane BH, the straight line AF is drawn perpendicular to that plane. Q. E. F.

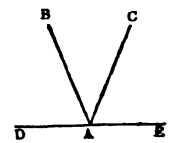


Cor. If it be required from a point C in a plane to erect a perpendicular to that plane, take a point A above the plane, and draw AF perpendicular to the plane; then, if from C a line be drawn parallel to AF, it will be the perpendicular required; for being parallel to AF it will be perpendicular to the same plane to which AF is perpendicular (7. Pl.).

PROP. XI. THEOR.

From the same point in a given plane, there cannot be two straight lines at right angles to the plane upon the same side of it: and there can be but one perpendicular to a plane from a point above the plane.

For, if it be possible, let the two straight lines AC, AB be at right angles to a given plane from the same point A in the plane, and upon the same side of it; and let a plane pass through BA, AC: the common section of this with the given plane is a straight (3. Pl.) line passing through A. Let DAE be their common section: therefore the straight lines AB, AC, DAE are in one plane; and because CA is at right angles to the given plane, it shall make right angles with every straight line meeting it in that plane. But DAE, which is in that plane, meets CA; therefore CAE is a right angle. For the same reason, BAE is a right angle: wherefore the angle CAE is equal to the angle BAE; and they are in one plane, which is impossible. Also, from a point above a plane, there can be but one perpendicular to that plane; for, if there could be two, they would be parallel (6. Pl.) to one another, which is absurd. Therefore, from the same point, &c. Q. E. D.

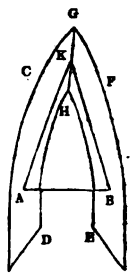


PROP. XII. THEOR.

PLANES to which the same straight line is perpendicular, are parallel to one another.

Let the straight line AB be perpendicular to each of the planes CD, EF: these planes are parallel to one another.

If not, they shall meet one another when produced; let them meet: their common section shall be a straight line GH, in which take any point K, and join A, K, and B, K. Then, because AB is perpendicular to the plane EF, it is perpendicular (1 Def. Pl.) to the straight line BK which is in that plane: therefore ABK is a right angle. For the same reason, BAK is a right angle; wherefore the two angles ABK, BAK of the triangle ABK are equal to two right angles, which is impossible (Euc. 17. 1.): therefore the planes CD, EF, though produced, do not meet one another: that is, they are parallel (7. Def. Pl.). Therefore, planes, &c. Q. E. D.



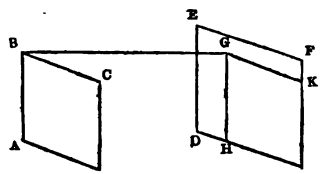
PROP. XIII. THEOR.

IF two straight lines meeting one another, be parallel to two straight lines which meet one another, but are

not in the same plane with the first two: the plane which passes through these is parallel to the plane passing through the others.

Let AB, BC, two straight lines meeting one another, be parallel to DE, EF that meet one another, but are not in the same plane with AB, BC: the planes through AB, BC, and DE, EF shall not meet, though produced.

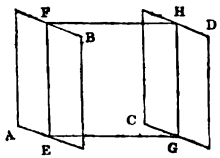
From the point B draw BG perpendicular (10. Pl.) to the plane which passes through DE, EF, and let it meet that plane in G, and through G draw GH parallel to ED (Euc. 31. 1.), and GK parallel to EF. And because BG is perpendicular to the plane through DE, EF, it shall make right angles with every straight line meeting it in that plane (1. Def. Pl.). But the straight lines GH, GK in that plane meet it; therefore each of the angles BGH, BGK is a right angle. And because BA is parallel (8. Pl.) to GH (for each of them is a parallel to DE), the angles GBA, BGH are together equal (Euc. 29. 1.) to two right angles: and BGH is a right angle; therefore also GBA is a right angle, and GB perpendicular to BA: for the same reason, GB is perpendicular to BC. Since, therefore, the straight line GB stands at right angles to the two straight lines BA, BC, that cut one another in B; GB is perpendicular (4. Pl.) to the plane through BA, BC: and it is perpendicular to the plane through DE, EF; therefore BG is perpendicular to each of the planes through AB, BC, and DE, EF: but planes to which the same straight line is perpendicular are parallel (12. Pl.) to one another: therefore the plane through AB, BC is parallel to the plane through DE, EF. Wherefore, if two straight lines, &c. Q. E. D.



PROP. XIV. THEOR.

If two parallel planes be cut by another plane, their common sections with it are parallel.

Let the parallel planes AB, CD be cut by the plane EFHG, and let their common sections with it be EF, GH: EF is parallel to GH.



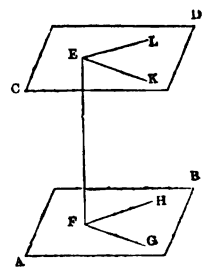
For the straight lines EF and GH are in the same plane, viz. EFHG, which cuts the planes AB and CD; and they do not meet though produced; for the planes in which they are do not meet; therefore EF and GH are parallel (Euc. 35. Def. 1.). Q. E. D.

PROP. XV. THEOR.

If a straight line be perpendicular to one of two parallel planes: it is perpendicular to the other.

Let the plane CD be parallel to the plane AB, and the straight line EF perpendicular to the plane AB: EF is also perpendicular to the plane CD.

Having made any two planes pass through EF, let one of them cut the parallel planes in the straight lines FG and EK, and let the other cut the same planes in the straight lines FH and EL. The plane CD, being parallel to the plane AB, EK (14. Pl.), is parallel, to FG, and EL to FH. And because EF is perpendicular to the plane AB, it is perpendicular to FG and to FH (1. Def. Pl.). But a straight line which is perpendicular to one of two parallel straight lines is perpendicular to the other: wherefore EF is perpendicular to each of the straight lines EK and EL; and consequently, it is perpendicular to the plane CD, in which EK and EL are situated (4. Pl.).



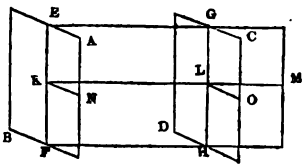
PROP. XVI. THEOR.

If two parallel planes be cut by a third plane: they have the same inclination to that plane.

Let AB and CD be two parallel planes, and EH a third plane cutting them: the planes AB and CD are equally inclined to EH.

Let the straight lines EF and GH be the common sections of the plane EH with the two planes AB and CD; and from K, any point in EF, draw in the plane EH the straight line KM at right angles to EF, and let it meet GH in L; draw also KN at right angles to EF in the plane AB; and through

the straight lines KM, KN, let a plane be made to pass, cutting the plane CD in the line LO. And because EF and GH are the common sections of the plane EH with the two parallel planes AB and CD, EF is parallel to GH (14. Pl.). But EF is at right angles to the plane that passes through KN and KM (4. Pl.), because it is at right angles to the lines KM and KN: therefore GH is also at right angles to the same plane (7. Pl.), and it is therefore at right angles to the lines LM, LO which it meets in that plane.



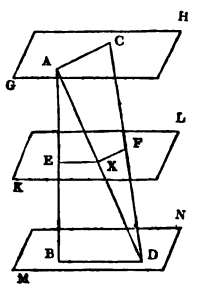
Therefore, since LM and LO are at right angles to LG, the common section of the two planes CD and EH, the angle OLM is the inclination of the plane CD to the plane EH (4. Def. Pl.). For the same reason, the angle MKN is the inclination of the plane AB to the plane EH. But because KN and LO are parallel, being the common sections of the parallel planes AB and CD with a third plane, the interior angle NKM is equal to the exterior angle OLM (Euc. 29. 1.): that is, the inclination of the plane AB to the plane EH is equal to the inclination of the plane CD to the same plane EH. Therefore, &c. Q. E. D.

PROP. XVII. THEOR.

If two straight lines be cut by parallel planes: they shall be cut in the same ratio.

Let the straight lines AB, CD be cut by the parallel planes GH, KL, MN, in the points A, E, B; C, F, D: as AE is to EB, so is CF to FD.

Draw AC, BD, AD, and let AD meet the plane KL in the point X; also draw EX, XF. Because the two parallel planes KL, MN are cut by the plane EBDX, the common sections EX, BD are parallel (14. Pl.). For the same reason, because the two parallel planes GH, KL are cut by the plane AXFC, the common sections AC, XF are parallel: and because EX is parallel to BD, a side of the triangle ABD, as AE to EB, so is (Euc. 2. 6.) AX to XD. Again, because XF is parallel to AC, a side of the triangle ADC, as AX to XD, so is CF to FD: and it was proved that AX is to XD, as AE to EB: therefore (5. Cor. Def. Pr.), as AE to EB, so is CF to FD. Wherefore, if two straight lines, &c. Q. E. D.

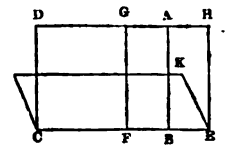


PROP. XVIII. THEOR.

If a straight line be at right angles to a plane: every plane which passes through it is also at right angles to that plane.

Let the straight line AB be at right angles to a plane CK: every plane which passes through AB is at right angles to the plane CK.

Let any plane DE pass through AB, and let CE be the common section of the planes DE, CK; take any point F in CE, from which draw FG in the plane DE at right angles to CE. And because AB is perpendicular to the plane CK, therefore it is also perpendicular to every straight line meeting it in that plane (1. Def. Pl.) and consequently it is perpendicular to CE: wherefore ABF is a right angle; but GFB is likewise a right angle; therefore AB is parallel (Euc. 28. 1.) to FG. And AB is at right angles to the plane CK; therefore FG is also at right angles to the same plane (7. Pl.). But one plane is at right angles to another plane when the straight lines drawn in one of the planes, at right angles to their common section, are also at right angles to the other plane (2. Def. Pl.); and any straight line FG in the plane DE, which is at right angles to CE, the common section of the planes, has been proved to be perpendicular to the other plane, CK; therefore the plane DE is at right angles to the plane CK. In like manner it may be proved that all the planes which pass through AB are at right angles to the plane CK. Therefore, if a straight line, &c. Q. E. D.



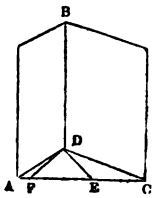
PROP. XIX. THEOR.

If two planes cutting one another be each of them perpendicular to a third plane: their common section shall be perpendicular to the same plane.

Let the two planes, AB, BC be each of them perpendicular to a third plane ADC, and let BD be the common section of the first two: BD is perpendicular to the plane ADC.

From D, in the plane ADC, draw DE perpendicular to AD, and DF to DC. Because DE is perpendicular to AD, the

common section of the planes AB and ADC; and because the plane AB is at right angles to ADC, DE is at right angles to the plane AB (2. Def. Pl.): and therefore also to the straight line BD in that plane (1. Def. Pl.). For the same reason, DF is at right angles to DB. Since BD is therefore at right angles to both the lines DE and DF, it is at right angles to the plane in which DE and DF are, that is, to the plane ADC (4. Pl.). Wherefore, &c. Q. E. D.

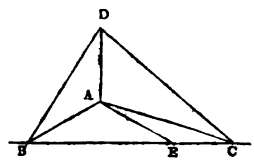


PROP. XX. THEOR.

If a solid angle be contained by three plane angles: any two of them are greater than the third.

Let the solid angle at A be contained by the three plane angles BAC, CAD, DAB: any two of them are greater than the third.

If the angles BAC, CAD, DAB be all equal, it is evident that any two of them are greater than the third. But if they are not, let BAC be that angle which is not less than either of the two others, and is greater than one of them DAB; and at the point A in the straight line AB make, in the plane which passes through BA, AC, the angle BAE equal (Euc. 23. 1.) to the angle DAB; and make AE equal to AD, and through E draw BEC, cutting AB, AC in the points B, C, and draw DB, DC. And because DA is equal to AE, and AB is common to the two triangles ABD, ABE, and also the angle DAB equal to the angle EAB; therefore the base DB is equal (Euc. 4. 1.) to the base BE. And because BD, DC are greater (Euc. 20. 1.) than CB, and one of them BD has been proved equal to BE, a part of CB, therefore the other DC is greater than the remaining part EC. And because DA is equal to AE, and AC common, but the base DC greater than the base EC, therefore the angle DAC is greater (Euc. 25. 1.) than the angle EAC; and, by the construction, the angle DAB is equal to the angle BAE; wherefore the angles DAB, DAC are together greater than BAE, EAC, that is, than the angle BAC. But BAC is not less than either of the angles DAB, DAC; therefore BAC, with either of them, is greater than the other. Wherefore, if a solid angle, &c. Q. E. D.

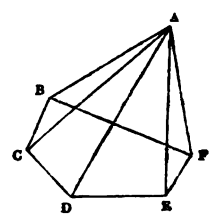


PROP. XXI. THEOR.

THE plane angles, which contain any solid angle, are together less than four right angles.

Let A be a solid angle contained by any number of plane angles BAC, CAD, DAE, EAF, FAB: these together are less than four right angles.

Let the planes which contain the solid angle at A be cut by another plane, and let the section of them by that plane be the rectilineal figure BCDEF. And because the solid angle at B is contained by three plane angles CBA, ABF, FBC of which any two are greater (20. Pl.) than the third, the angles CBA, ABF are greater than the angle FBC: for the same reason, the two plane angles at each of the points C, D, E, F, viz. the angles which are at the bases of the triangles having the common vertex A, are greater than the third angle at the same point, which is one of the angles of the figure BCDEF: therefore all the angles at the bases of the triangles are together greater than all the angles of the figure: and because all the angles of the triangles are together equal to twice as many right angles as there are triangles (Euc. 32. 1.); that is, as there are sides in the figure BCDEF; and because all the angles of the figure, together with four right angles, are likewise equal to twice as many right angles as there are sides in the figure (Euc. 1. Cor. 32. 1.); therefore all the angles of the triangles are equal to all the angles of the rectilineal figure, together with four right angles. But all the angles at the bases of the triangles are greater than all the angles of the rectilineal figure, as has been proved. Wherefore, the remaining angles of the triangle, viz. those at the vertex, which contain the solid angle at A, are less than four right angles. Therefore, every solid angle, &c. Q. E. D.



SCHOLIUM.

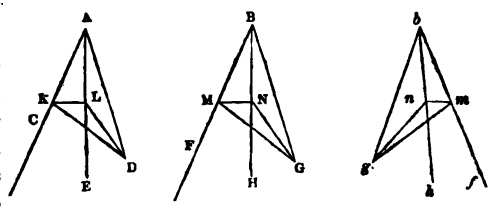
This demonstration proceeds upon the supposition that none of the planes which contain the solid angle, make re-entering angles with one another, or pass within the solid angle in such a manner as, when produced, to divide it. Without this limitation, the proposition is not true.

PROP. XXII. THEOR.

If each of two solid angles be contained by three plane angles that are equal to one another, each to each: the planes of the equal angles have equal inclinations to one another.

Let there be two solid angles at the points A and B, each of which is contained by three plane angles that are equal to one another, each to each, viz. the angle CAE to FBH, CAD to FBG, and EAD to HBG: the planes of the equal angles have equal inclinations to one another.

In AC take any point K, and draw KL in the plane CAE, and KD in the plane CAD, both perpendicular to CA; and join L, D. Take BM equal to AK, and from the point M draw in the planes FBH, FBG, the lines, MN, MG at right angles to BF; and join N, G. In the triangles KAL and MBN, the angle KAL is equal to MBN, and the right angle AKL to the right angle BMN, and AK to BM; therefore AL is equal to BN; and KL to MN (Euc. 26. 1.). For the same reason, in the triangles AKD and BMG, AD is equal to BG, and KD to MG. And, because AL is equal to BN, and AD to BG, and the angle LAD to NBG, the triangles LAD and NBG are equal (Euc. 4. 1.), and consequently LD is equal to NG. Lastly, in the triangles LKD and NMG, KL is equal to MN, and KD to MG, and LD to NG; therefore the angle LKD is equal to the angle NMG (Euc. 8. 1.). But the angle LKD is the inclination of the plane CAE to the plane CAD (4. Def. Pl.), and the angle NMG is the inclination of the plane FBH to the plane FBG, which planes are therefore equally inclined to one another. And in the same manner it may be proved that the other planes containing the equal angles have equal inclinations to one another. Therefore, &c. Q. E. D.



The equal plane angles of the two solid angles may be disposed in the same order as in the figure referred to in the demonstration, or they may be disposed in a contrary order as represented in the solid angles at A and *b*; the plane angle CAE of the solid angle at A being equal to the plane angle

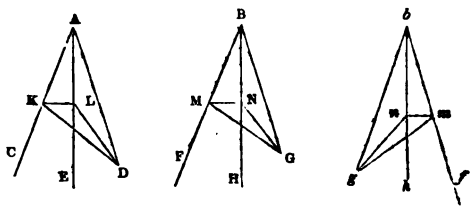
fbh of the solid angle at *b*; and EAD to *hbg*, and CAD to *fbg*. But this distinction does not affect the demonstration.

PROP. XXIII. THEOR.

If two solid angles, each of which is contained by three plane angles, have two of their plane angles equal, each to each, and have likewise the planes in which those angles are, alike inclined to one another: their remaining plane angles shall be equal: and the planes in which they are shall have like inclinations to the other planes of the solid angles.

Let the two solid angles at the points A and B, each of which is contained by three plane angles, have two of their plane angles equal each to each, viz. the angle CAD to FBG, and CAE to FBH; and likewise let the plane CAE have the same inclination to the plane CAD, that the plane FBH has to the plane FBG: the remaining angle EAD shall be equal to the remaining angle HBG; and the plane EAD shall have the same inclinations to the planes CAD and CAE, that the plane HBG has to the planes FBG and FBH.

In AC take any point K, and draw KL in the plane CAE, and KD in the plane CAD, both perpendicular to CA; and join L, D. Take BM equal to AK, and from the point M draw in the planes FBH, FBG the lines MN, MG at right angles to BF; and join N, G. In the triangles KAL and MBN the angle KAL is equal to MBN, and the right angle AKL to the right angle BMN, and AK to BM; therefore AL is equal to BN, and KL to MN (Euc. 26. 1.). For the same reason in the triangles AKD and BMG, AD is equal to BG, and KD to MG. But the angle LKD is the inclination of the plane CAE to the plane CAD (4. Def. Pl.); and the angle NMG is the inclination of the plane FBH to the plane FBG; therefore the angle LKD is equal to the angle NMG. And, because KL is equal to NM, and KD to MG, and the angle LKD to the angle NMG, the triangles LKD and NMG are equal, and LD is equal (Euc. 4. 1.) to NG. And, in the triangles LAD and NBG, AL is equal to BN, AD to BG, and



LD to NG; therefore the angle EAD is equal to the angle HBG (Euc. 8. 1.). Again, because the plane angles which contain the solid angle at A, are equal to the plane angles which contain the solid angle at B, each to each, the plane EAD has the same inclinations to the planes CAD and CAE, that the plane HBG has to the planes FBG and FBH (22. Pl.). Therefore, &c. Q. E. D.

ELEMENTS

OF

GEOMETRY.

SOLIDS.

PRISMS AND PYRAMIDS.

DEFINITIONS.

I.

A SOLID is that which has length, breadth, and thickness.

II.

A prism is a solid figure contained by plane figures of which two, that are opposite, are equal, similar, and parallel to one another; and the rest are parallelograms.

III.

The altitude of a prism is the perpendicular distance of the two equal and similar plane figures between which the solid is constituted.

IV.

The length of a prism is equal to the length of its lateral edges.

V.

A prism is upright when all the lateral parallelograms are perpendicular to the equal and parallel ends; otherwise it is oblique.

VI.

A parallelepiped is a solid contained by six quadrilateral figures, of which every two that are opposite are parallel.

VII.

A parallelepiped is rectangular when all the intersecting planes are perpendicular to one another; otherwise it is oblique.

VIII.

A cube is a solid figure contained by six equal squares.

IX.

A pyramid is a solid figure bounded, on its sides, by plane rectilineal triangles that are constituted between any rectilineal figure and a point above it, which is the common vertex of all the triangles.

The plane rectilineal figure on which the pyramid is constituted is called the base of the pyramid; and the point above the base where all the triangles meet, is called the vertex of the pyramid.

X.

The altitude of a pyramid is the perpendicular drawn from the vertex upon the plane of the base.

XI.

Similar solid figures are such as are contained by the same number of similar planes, similarly situated, and having like inclinations to one another.

XII.

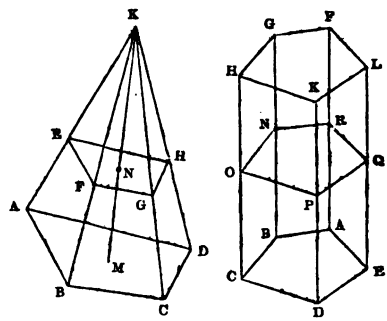
Two solid figures which are bounded by the same number of similar and equal planes, having equal inclinations to one another, but having their angles and homologous sides disposed in contrary order, are called symmetrical solids. Such solids, though they have their several parts respectively equal to one another, will not coincide.

N. B. Solid figures which are bounded by any number of plane surfaces, except such as are denominated Prisms or Pyramids, are called Polyhedrons.

PROP. I. THEOR.

A SECTION of a pyramid made by a plane parallel to the base is a figure similar to the base: and a section of a prism made by a plane parallel to the base is a figure similar and equal to the base.

Let the pyramid K-ABCD be cut by a plane parallel to the base ABCD: the section EFGH is similar to the base. Because the plane AKB cuts two parallel planes, the common sections AB and EF are parallel (14. Pl.): for the same reason, EH is parallel to AD. Since therefore EF and EH are parallel to AB and AD, although not in the same plane with them, the angle FEH is equal to the angle BAD (9. Pl.). In like manner the other angles of the figures EFGH and ABCD are proved to be equal, each to each. Again, because the triangle KEF is similar to KAB, and the triangle KEH to KAD,



FE : EK :: BA : AK

EK : EH :: KA : AD

ex æquali,

FE : EH :: BA : AD.

And, in like manner, it may be proved that the sides about the other angles of the figures EFGH and ABCD are proportionals. Wherefore, since the figures have been shown to be equiangular, they are similar.

Next, let the prism ABCDE-FGHKL be cut by a plane parallel to the base; the section NOPQR is similar and equal to the base. For OP and CD are parallel, because they are the common sections of the plane KC with two parallel planes (14. Pl.); and OC is parallel to DP: wherefore OP is equal and parallel to DC (Euc. 34. 1.). In like manner it is proved that the other sides of the section are equal and parallel to the other sides of the base, each to each. Because the sides of the section are respectively parallel to the sides of the base, the angles of the section are equal to the angles of the base, each to each (9. Pl.): and the sides of the two figures are likewise equal, each to each: wherefore the sections are similar and equal to one another. Q. E. D.

Cor. The base of a pyramid is to any section parallel to it in the duplicate ratio of their distances from the vertex of the pyramid.

Let EFGH be a section parallel to the base of the pyramid K-ABCD and from K, draw KM perpendicular to the plane of the base, cutting the figure EFGH in N: then KM and KN will be the distances of the base and the section from the vertex K.

Because the figure ABCD is similar to EFGH, they are to one another in the duplicate ratio of their homologous sides (Euc. 20. 6.); that is,

$$ABCD : EFGH :: AB^2 : EF^2.$$

Again, the triangle AKB being similar to EKF,

$$AB : EF :: KA : KE \text{ (Euc. 4. 6.)};$$

hence $AB^2 : EF^2 :: KA^2 : KE^2$ (Euc. 22. 6.).

But the parallel planes ABCD, EFGH cut the lines KA, KM proportionally in E and N (17. Planes.); therefore

$$KA : KE :: KM : KN, \text{ and}$$

$$KA^2 : KE^2 :: KM^2 : KN^2.$$

Consequently, by equality of ratios,

$$ABCD : EFGH :: KM^2 : KN^2;$$

that is the base of the pyramid, and any section parallel to that base, are to one another in the duplicate ratio of their distances from the vertex.

PROP. II. THEOR.

Prisms of equal lengths, that have all their lateral edges in the same straight lines, are equal or equivalent to one another.

Let ABCD-EFGH and KLMN-OPQR be two prisms that have their lateral edges, viz. AE, BF, CG, DH, KO, LP, MQ, NR; of the same length, and in the same straight lines: the prisms are equal or equivalent to one another.

Because AE is equal to KO, and KE common, AK is equal to EO. For a like reason, BL is equal to FP, and CM to GQ, and DN to HR. And because (Euc. 34. 1.) AB is equal to EF, and LK to PO, the sides of the figure ABLK are equal to the sides of the figure EFPO, each to each. The angles of the same figures, contained by their equal sides, are also equal, each to each; for the angle AKL is equal to EOP (Euc. 29. 1.), and KLB to OPF, and LBA to PFE, and BAK to FEO: therefore the figure AKLB is similar and equal to the figure EOPF: and in the same manner it may be proved that the

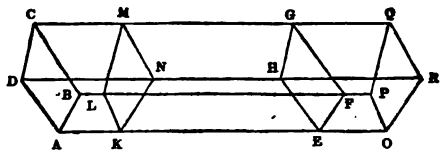


figure AKND is similar and equal to EORH, and DCMN to HGQR, and BCML to FGQP. Again because the plane AP meets the parallel planes KLMN and OPQR, it is equally inclined to both (16. Pl.) that is, the plane AL has the same inclination to KLMN, that the plane EP has to OPQR. And in the same manner it is proved that the planes AN, NC, BM have respectively the same inclinations to KLMN, that the planes ER, RG, FQ have to the plane OPQR. Let the solid OPQR-EFGH be applied to the solid KLMN-ABCD, so that the planes OPQR and KLMN, which are equal and similar may coincide, the point O being upon K, P upon L, and so on: then, because the inclination of EP to OPQR is equal to the inclination of AL to KLMN, the plane EP will be upon the plane AL, and will coincide with it, because they are equal and similar, and because their equal sides OP and KL coincide. And in the same manner it is proved that the plane AN will coincide with ER, and DM with HQ, and BM with FQ. Therefore the remaining plane ABCD will coincide with the remaining plane EFGH, and the two solids will wholly coincide, and be similar and equal. And if each of the equal solids be taken from the whole solid ABCD-OPQR, the remainders will be equal or equivalent, that is, the prism ABCD-EFGH is equal or equivalent to the prism KLMN-OPQR. Q. E. D.

PROP. III. THEOR.

In a parallelepiped the opposite planes are similar and equal parallelograms.

Let AG be a parallelepiped: its opposite planes are similar and equal parallelograms.

Because the planes AC and EG are parallel, AB and EF, the common sections of those planes with the plane AF, are parallel (14. Pl.). And because the planes AH and BG are parallel, AE and BF, the common sections of those planes with the plane AF, are parallel (14. Pl.). Wherefore AF is a parallelogram: and, in like manner, it may be proved that each of the figures, DE, DG, CF, AC, EG is a parallelogram. Again, because DE, DG, CF, AF are parallelograms, the sides of the

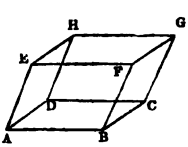


figure EG are parallel to the sides of the figure AC, each to each (Euc. 34. 1.): wherefore the figures EG and AC are equiangular (9. Pl.). But the sides of the figure EG are also equal to the sides of the figure AC, each to each (Euc. 34. 1.): wherefore the figures EG and AC are similar and equal. And in the same manner it may be shown that the parallelogram AF is similar and equal to DG, and DE to CF.

Cor. 1. In a rectangular parallelepiped, the opposite planes are rectangles.

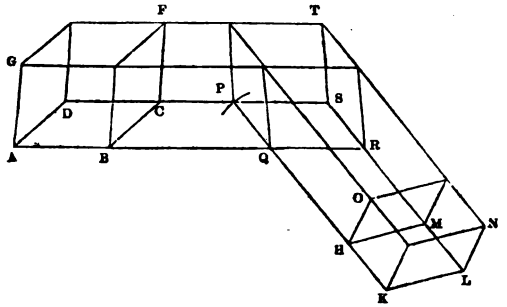
Cor. 2. A parallelepiped is a prism, whichsoever of the six parallelograms that bound it, is taken for the base.

PROP. IV. THEOR.

SOLID parallelepipeds, which are upon equal or equivalent bases, and of equal altitudes, are equal or equivalent to one another.

Let the solid parallelepipeds AF and HN be upon equal or equivalent bases AC and HL, and likewise have equal altitudes: the solid AF is equal or equivalent to the solid HN.

Of the two straight lines AB and HM, let HM be that which is not greater than AB: and, because the parallelogram HL is equal to the parallelogram AC, the perpendicular distance of the parallel lines HM and KL is not less than the perpendicular distance of the parallel lines AB and DC (Euc. 36. 1.): wherefore the straight line HK is not less than the perpendicular distance of the same parallels. Produce AB and DC to R and S, and take QR equal to AB: and because HK is not less than the perpendicular distance of the parallel lines AB and DC, a circle described from the centre Q with the distance



HK, will cut, or touch, the straight line DS; let the circle cut, or touch, DS in P, draw QP, and produce it to K, and

draw RS parallel to QP. Let the solid HN be so placed that HK may be upon the straight line PK, and the base HL upon the same plane with the base AC, and on the same side of the line PK with the parallelogram QS. And, because the parallelogram AC is equal or equivalent to QS (Euc. 36. 1.), and likewise to HL, therefore QS is equal or equivalent to HL: and they are upon equal bases in the same straight line; they are therefore between the same parallels, and have their sides SR and ML situated in one straight line (Euc. 36. 1.). Again, because the solids AF and HN have equal altitudes, and their bases are upon one plane, their planes GF and ON opposite to the bases, are likewise upon one plane. Now the six planes following, which are parallel two and two, viz. the two planes between which the solids AF and HN are situated, the two planes BG and DF of the solid AF, and the two planes KO and MN of the solid HN, being produced, will intersect one another, and will form a parallelepiped QT upon the base QRSP (6. Def. Prisms, &c.). The parallelepiped, or prism (2. Cor. 3. Prisms, &c.), AF is equal or equivalent to the parallelepiped, or prism, QT, because their lateral edges are of equal length, and in the same straight lines (2. Prisms, &c.). For the same reasons the parallelepiped, or prism HN is equal or equivalent to the parallelepiped, or prism QT. Wherefore the solid AF is equal or equivalent to the solid HN. Q. E. D.

COR. From this proposition, and from A. Pr., it follows that a parallelepiped having a given altitude and a base less than any plane surface that can be assigned, is a magnitude less than any solid that can be assigned: for it is equivalent to that which remains when from a given parallelepiped (having an equal altitude), there is taken one which, having the same altitude, is equal to its half or more than its half; from the remainder one which, having the same altitude, is equal to half or more than half of that remainder; and so on.

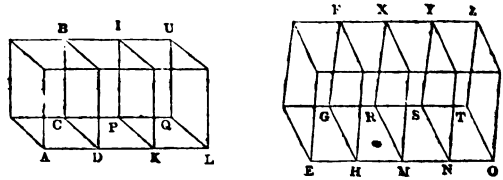
PROP. V. THEOR.

PARALLELEPIPEDS that have equal altitudes are to one another as their bases.

Let AB and EF be parallelepipeds that have equal altitudes: as the base AC is to the base EG, so is the solid AB to the solid EF.

Produce AD and EH, and take any number of straight lines DK, KL, each equal to AD; and any number HM, MN,

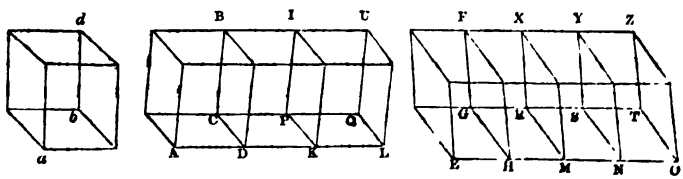
NO each equal to EH: complete the parallelograms DP, KQ,



HR, MS, NT, and the solids DI, KU, HX, MY, NZ. Because AD, DK, KL are equal, the parallelograms AC, DP, KQ are equal or equivalent (Euc. 36. 1.): wherefore the solids AB, DI, KU are equal or equivalent to one another (4. Prisms, &c.). It is manifest therefore that the solid AU is the same multiple of the solid AB, that the parallelogram AQ is of the parallelogram AC. And in like manner it may be shown that the solid EZ is the same multiple of EF that the parallelogram ET is of EG. Now, if the parallelogram AQ be equal or equivalent to the parallelogram ET, the solid AU is equal or equivalent to the solid EZ, because they have equal altitudes (4. Prisms, &c.); and if the parallelogram AQ be greater than ET, AU is greater than EZ; and if less, less. Wherefore, as the base AC is to the base EG, so is the solid AB to the solid EF (5. Def. 5. Euclid, or 10. Def. Pr.). Q. E. D.

Otherwise,

Let the parallelepipeds AU and EZ have equal altitudes: as the base AQ to the base ET, so is the solid AU to the solid EZ. Suppose that the bases AQ and ET have a common

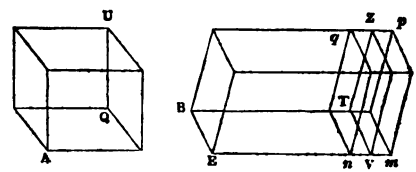


measure, viz. the parallelogram ab; and upon ab construct the parallelepiped ad, having the same altitude with the solids AU and EZ. Cut off AD the same part of AL, that ab is of AQ (Euc. 9. 6.); make DK, KL each equal to AD, and complete the parallelograms AC, DP, KQ, and the solids AB, DI, KU. In like manner, cut off EH the same part of EO, that ab is of ET; make HM, MN, NO each equal to EH, and complete the parallelograms EG, HR, MS, NT, and the solids EF, HX, MY, NZ. Because AD, DK, KL are all equal, the parallelograms AC, DP, KQ are all equal or equivalent (Euc.

36. 1.); wherefore the parallelogram AC is the same part of AQ, that AD is of AL, or the parallelogram *ab* of AQ. The parallelogram *ab* is therefore equal or equivalent to the parallelogram AC, and to each of the parallelograms DP, KQ. And because parallelepipeds that have equal bases and equal altitudes are equal or equivalent (4. Prisms, &c.), the solid *ad* is equal or equivalent to each of the solids AB, DI, KU. In the same manner it is proved that the parallelogram *ab* is equal or equivalent to each of the parallelograms EG, HR, MS, NT, and the solid *ad*, equal or equivalent to each of the solids EF, HX, MY, NZ. Therefore, whatever part the parallelogram *ab* is of the base ET, the same part is the solid *ad* of the solid EZ; and as often as the base AQ contains the parallelogram *ab*, so often does the solid AU contain the solid *ad*: consequently,

AQ : ET :: solid AU : solid EZ (10. Def. Pr.).

If the bases AQ and ET are incommensurable, by continually bisecting one of them, as AQ, there may at length (1. Cor. A. Pr.) be found a surface which is equal to, or less than, the least that can be assigned: let this surface be equivalent to a parallelogram whose altitude is equal to the distance between the lines BT and EV, and whose base is less than any assignable line (4. Cor. A. Pr.); then such parallelogram will (D. Pr.) be contained in ET a certain number of times with a remainder, as *n*T, or an excess, as T*m*, less than itself; that is less than any surface that can be assigned. Complete the parallelepipeds V*q* and V*p*: then the bases AQ and B*n*, AQ and B*m* being commensurable,



AU : E*q* :: AQ : B*n*,

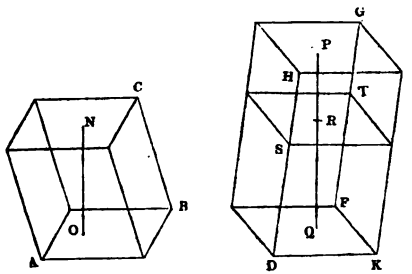
and AU : E*p* :: AQ : B*m*;

also the difference between B*n* and B*m* is equal to, or less than, the least surface that can be assigned, and (Cor. 4. Prisms.) the difference between E*q* and E*p* is equal to, or less than, the least solid that can be assigned; therefore, by the Second General Theorem, (Propor.) AU : EZ :: AQ : ET.

PROP. VI. THEOR.

SOLID parallelepipeds that have equal or equivalent bases, are to one another as their altitudes.

Let AC and DG be parallelepipeds upon the equal or equivalent bases AB and DF; and let NO and PQ be perpendiculars let fall upon the planes of the bases from any points N and P in the planes opposite to them: then NO and PQ are the altitudes of the solids AC and DG: as the solid AC is to the solid DG so is the altitude NO to the altitude PQ.



Take QR equal to ON, and through R let a plane be made to pass parallel to the plane HG so as to form, with the other planes of the solid DG, the parallelepiped DT. If the solids DT and DG be conceived to stand upon the bases SK and HK, they have the same altitude: therefore DT : DG :: SK : HK (5. Prisms, &c.) :: DS : DH (Euc. 1. 6.); and DS : DH :: QR : QP (17. Planes.); therefore DT : DG :: QR : QP. But QR is equal to ON; and the solid DT is equal or equivalent to the solid AC, because they are upon equal or equivalent bases DF and AB, and have equal altitudes QR and ON (4. Prisms, &c.): wherefore as the solid AC is to the solid DG, so is the altitude ON to the altitude QP. Q.E.D.

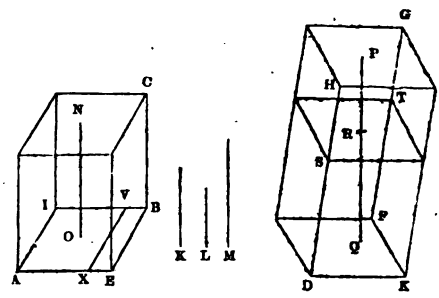
PROP. VII. THEOR.

SOLID parallelepipeds are to one another in the ratio which is compounded of the ratios of the areas of their bases, and of their altitudes.

Let AC and DG be two parallelepipeds, of which the bases are the parallelograms AB and DF, and the altitudes, the straight lines NO and PQ drawn perpendicular to the planes of the bases from any points in the planes opposite to them: the solid AC has to the solid DG the ratio which is compounded of the ratio of the base AB to the base DF, and of the altitude NO to the altitude PQ.

To the straight line AI apply the parallelogram AV equal or equivalent to DF, and having IAX for one of its angles (Euc. Cor. 45. 1.). Taking any straight line K, make as AE to AX, so K to L; and as NO to PQ, so L to M. In QP take QR equal to NO, and through R let a plane pass parallel to

HG, forming, with the other planes of the solid DG, the parallelepiped DT. Because the parallelogram AV is equal or equivalent to DF, AB is to DF, as AB to AV, that is, as AE to AX (Euc. 1. 6.): but AE is to AX, as K to L; therefore AB is to DF, as K to L. And NO is to PQ, as L to M. But the ratio of K to M is said to be compounded of the ratios of K to L and of L to M; wherefore the ratio of K to M is that which is compounded of the ratios of the base AB to the base DF, and of the altitude NO to the altitude PQ.



Because the solids AC and DT have equal altitudes,

AC : DT :: AB : DF (5. Prisms, &c.)

but AB : DF :: K : L;

therefore AC : DT :: K : L. Again, because the solids DT and DG have the same base,

DT : DG :: QR : QP (6. Prisms, &c.)

but QR, or NO : QP :: L : M;

therefore DT : DG :: L : M. And since it has been proved that AC is to DT, as K to L; and DT to DG, as L to M; ex æquali, AC is to DG, as K to M.

Wherefore the solid AC has to the solid DG the ratio which is compounded of the ratios of the base AB to the base DF, and of the altitude NO to the altitude PQ. Q. E. D.

Cor. Parallelepipeds are to one another as the products of any numbers which are proportional to the areas of their bases, and to their altitudes.

Let AC and DG be parallelepipeds, and let the numbers B and *b* be proportional to the areas of the bases AB and DF, and the numbers A and *a* proportional to the altitudes NO and PQ: the solid AC is to the solid DG, as B × A to *b* × *a*.

For, AC : DT (:: AB : DF) :: B : *b* :: B.A : *b*.A.

Again, DT : DG (:: NO : QP) :: A : *a* :: A.*b* : *a*.*b*.

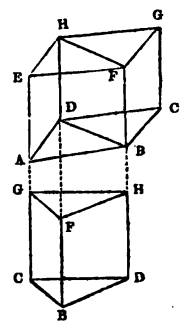
Therefore, ex æquali, AC : DG :: B.A : *b*.*a*.

PROP. VIII. THEOR.

If a parallelepiped be cut by a diagonal plane which passes through two of its opposite edges: it shall be cut into two equal or equivalent prisms.

Let ABCD-EFGH be a parallelepiped. The two opposite edges DH and BF being, each, parallel and equal to AE, they are parallel and equal to one another (8. Planes.). The figure BFHD is, therefore, a parallelogram (Euc. 33. 1.), and it divides the solid into two triangular prisms: these prisms are equal or equivalent to one another.

Because the three parallelograms which meet at the point G are respectively equal and similar to the three parallelograms which meet at A (3. Prisms, &c.), it is manifest that the plane angles which contain the solid angle at G are equal to the plane angles which contain the solid angle at A, each to each, viz. the angle FGH to DAB, FGC to DAE, and CGH to BAE: wherefore the inclination of the planes GB and GD is equal to the inclination of the planes AH and AF (22. Planes.). Let one of the prisms, GFH-CBD, be applied below the other, so that the parallelogram BG may be in the same plane with AH, and GC in the same straight line with AE; then, the parallelogram GD will be in the same plane with AF, because it was proved that the inclination of the planes GB and GD is equal to the inclination of the planes AH and AF. And because the parallelograms GB and AH are equal, and likewise stand upon equal bases situated in one straight line, they are between the same parallels (Euc. 36. 1.): wherefore HD and FB are in one straight line. In like manner, because the parallelograms GD and AF are equal, FB and HD are in one straight line. And, because the two prisms have equal lengths, and it is proved that their lateral edges are in the same straight lines, they are equal or equivalent to one another (2. Prisms, &c.). Q. E. D.



Cor. A triangular prism ABD-EFH is equivalent to a parallelepiped that has an equal base and an equal altitude.

Complete the parallelogram ABCD, and the parallelepiped AG. A parallelepiped that has its base equal to the triangle ABD, and the same altitude with the prism, is equal or equivalent to half the parallelepiped AG (5. Prisms, &c.), and it is, therefore equivalent to the prism ABD-EFH.

SCHOLIUM.

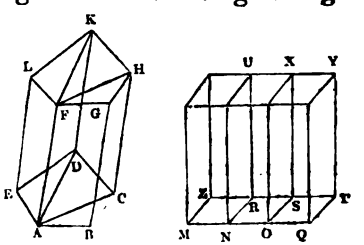
If the parallelepiped is not rectangular, the two triangular prisms into which it is divided are symmetrical solids, which cannot be brought to coincide: and the demonstration of this proposition may be applied to prove that any two symmetrical prisms are equivalent to one another.

PROP. IX. THEOR.

ANY prism is equivalent to a parallelepiped that has an equal base and an equal altitude.

Let the rectilineal figure ABCDE be equivalent to the parallelogram MT; a prism upon the base ABCDE is equivalent to a parallelepiped upon the base MT, and having an altitude equal to that of the prism.

Draw AD, AC, FK, FH. Because AF is equal and parallel to each of the lines DK and CH, the figures AFKD and AFHC are parallelograms (Euc. 33. 1.). The whole prism is therefore divided into triangular prisms, viz. those standing upon the triangles AED, ADC, ABC. To ZM apply the parallelogram MR equivalent to the triangle AED, and having an angle equal to ZMN (Euc. 44. 1.); and to NR apply the parallelogram NS equivalent to the triangle ADC and having an angle equal to RNO; and the remaining parallelogram OT will be equivalent to the remaining triangle ABC. Complete the parallepipeds MU, NX. The solid MU is equivalent to the prism upon the triangle AED, because they have equal bases and altitudes (Cor. 8. Prisms, &c.); and in like manner, the solid NX is equivalent to the prism upon the triangle ADC; and the solid OY, to the prism upon the triangle ABC. Wherefore the whole parallelepiped is equivalent to the whole prism upon the base ABCDE. Q. E. D.



COR. 1. Prisms which have equal or equivalent bases and equal altitudes are equal to one another.

COR. 2. Prisms of equal altitudes are to one another as their bases (5. Prisms, &c.). And prisms that have equal or equivalent bases are to one another as their altitudes (6. Prisms, &c.).

COR. 3. All prisms are to one another in the ratio compounded of the ratios of their bases and altitudes (7. Prisms, &c.).

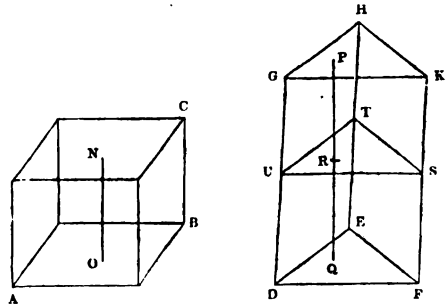
PROP. X. THEOR.

PRISMS, which have their bases and altitudes reciprocally proportional, are equivalent; and prisms which

are equivalent, have their bases and altitudes reciprocally proportional.

Let AC and DEF-GHK be two prisms, of which the bases are the figures AB and DEF, and the altitudes NO and PQ; and let the figure AB be to the figure DEF as PQ to NO; the prisms AC and DEF-GHK are equivalent.

In QP take QR equal to NO, and through R let a plane pass parallel to the plane GHK, so as to form the prism DEF-UTS, having its altitude QR equal to that of the prism AC. Because the prisms AC and DEF-UTS have equal altitudes, the prism AC is to the prism DEF-UTS as the base AB to the base DEF (2. Cor. 9. Prisms, &c.). And because the prisms DEF-GHK and DEF-UTS have the same base, the prism DEF-GHK is to the prism DEF-UTS as the altitude PQ to the altitude QR (2. Cor. 9. Prisms, &c.). But, by the hypothesis, the figure AB is to the figure DEF as PQ to QR, or NO: wherefore the prism AC is to the prism DEF-UTS as the prism DEF-GHK to the prism DEF-UTS. The prism AC is therefore equivalent to the prism DEF-GHK.



Again, if the prism AC be equivalent to the prism DEF-GHK, the base AB is to the base DEF as the altitude PQ to the altitude NO. For, the prisms AC and DEF-GHK being equivalent, the prism AC is to DEF-UTS, as the prism DEF-GHK to DEF-UTS. But because the prisms AC and DEF-UTS have equal altitudes, the prism AC is to the prism DEF-UTS as the base AB to the base DEF (2. Cor. 9. Prisms, &c.); and because the prisms DEF-GHK and DEF-UTS have the same base, the prism DEF-GHK is to the prism DEF-UTS as the altitude QP to the altitude QR, or NO (2. Cor. 9. Prisms, &c.). Wherefore the base AB is to the base DEF as the altitude PQ to the altitude NO. Q. E. D.

PROP. XI. THEOR.

THE convex surface of an upright prism is equal to a rectangle contained by the axis of the prism and a straight line equal to the perimeter of its base.

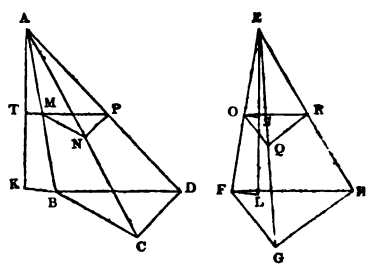
For each lateral plane on the surface of the prism is a rectangle contained by one of the lateral edges of the prism, and a side of the polygon which constitutes the base, and each lateral edge is equal to the axis: therefore, the sum of the lateral planes, that is the convex surface of the prism, is equal to the sum of all the rectangles contained by the axis and by the sides of the base; or the convex surface of the prism is equal to a rectangle contained by the axis and a line equal to the perimeter of the base. Q. E. D.

Cor. Hence the convex surfaces of upright prisms having equal altitudes, and whose bases are regular polygons of an equal number of sides, are to one another, as the perimeters of their bases; and if the prisms have equal bases, their convex surfaces are to one another as their altitudes.

PROP. XII. THEOR.

If two pyramids which have equal or equivalent bases, and equal altitudes, be cut by planes that are parallel to the bases, and at equal distances from them, the sections shall be equal or equivalent to one another.

Let A-BCD and E-FGH be two pyramids that have equal or equivalent bases BCD and FGH, and equal altitudes, viz. the lines AK and EL drawn from A and E, perpendicular upon the planes BCD and FGH: and let the pyramids be cut by the planes MNP and OQR parallel to BCD and FGH, and at equal distances KT and LS above those planes; the sections MNP and OQR are equal or equivalent.



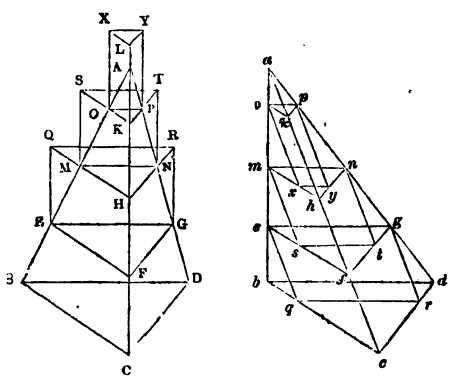
For BCD : MNP :: AK^2 : AT^2, and FGH : OQR :: EL^2 : ES^2 (Cor. 1. Prisms, &c.). But AK is equal to EL, and AT, to ES; therefore, by equality of ratios, BCD : MNP :: FGH : OQR, or by alternation, BCD : FGH :: MNP : OQR: but BCD is equal or equivalent to FGH; therefore MNP is equal or equivalent to OQR. Consequently, if two pyramids, &c. Q. E. D.

PROP. XIII. THEOR.

A SECTION of a pyramid made by a plane parallel to the base, how near soever the section may be to the base, will cut off a pyramid which is less than any pyramid that has its base and altitude equal or equivalent to the base and altitude of the first pyramid.

Let the pyramid A-BCD be cut by a plane parallel to the base BCD, which makes the section EFG: the pyramid A-EFG cut off from A-BCD, how near soever the section EFG is to the base BCD, will be less than any pyramid *a-bcd* that has its altitude equal to that of A-BCD, and its base *bcd* equal or equivalent to BCD.

Make FH, HK, KL each equal to CF, till the point L either falls exactly upon the vertex A, which will be the case when CF is contained a number of times, exactly, in CA; or till the point L falls above the vertex A: through H and K let planes be made to pass parallel to the base BCD, cutting the pyramid in the sections MHN, OKP; and upon the triangles EFG, MHN, OKP construct the prisms EFG-QHR, MHN-SKT; OKP-XLY. Let the pyramid *a-bcd* be cut by planes parallel to the base *bcd*, making the sections *efg*, *mhn*, *okp*, which have their altitudes above the base *bcd* respectively equal to the altitudes of the sections EFG, MHN, OKP above the base BCD: and upon the triangles *efg*, *mhn*, *okp*, construct the prisms *efg-qcr*, *mhn-sft*, *okp-xhy*. Because the pyramids A-BCD and *a-bcd* have equal or equivalent bases and altitudes, their sections made at equal distances above the bases are equal or equivalent (12. Prisms, &c.), viz. the section EFG to *efg*, and MHN to *mhn*, and OKP to *okp*: wherefore the prisms that stand upon the equal or equivalent sections, and which have equal altitudes, are equal or equivalent (1. Cor. 9. Prisms, &c.); that is, the prism EFG-QHR is equal to *efg-qcr*, and MHN-SKT to *mhn-sft*, and OKP-XLY to *okp-xhy*: wherefore the



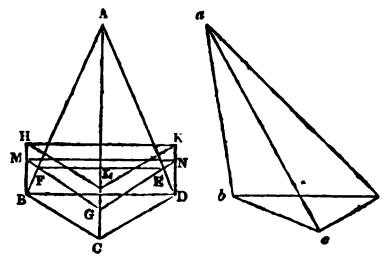
sum of all the prisms circumscribed about the pyramid A-EFG is equal or equivalent to the sum of all the prisms contained within *a-bcd*. But the pyramid *a-bcd* is greater than the sum of the prisms contained within it; and because the point L either coincides with the vertex A, or is above it, the sum of the prisms circumscribed about the pyramid A-EFG is greater than that pyramid: wherefore the pyramid *a-bcd* is greater than the pyramid A-EFG. Q. E. D.

PROP. XIV. THEOR.

PYRAMIDS that have equal or equivalent bases and equal altitudes are equal or equivalent to one another.

Let A-BCD and *a-bcd* be two pyramids that have equal or equivalent bases BCD and *bcd*, and likewise equal altitudes, viz. the perpendiculars drawn from the vertices A and *a* upon the planes BCD and *bcd*: the pyramid A-BCD is equal or equivalent to the pyramid *a-bcd*.

If they are not equal or equivalent, let A-BCD be greater than *a-bcd*: and let the excess of ABCD above *a-bcd* be equivalent to a prism BCD-HLK, upon the base BCD. Let a plane parallel to BCD pass between the points C and L, and cut the pyramid and prism in the sections EGF and MGN. The excess of the pyramid A-BCD above the pyramid A-EFG is less than the prism BCD-MGN, and, much more is it less than the prism BCD-HLK, that is, than the excess of the pyramid A-BCD above the pyramid *a-bcd*: wherefore, the pyramid A-EFG is greater than the pyramid *a-bcd*. But A-EFG is also less than the same pyramid (13. Prisms, &c.); which is impossible: wherefore the pyramids A-BCD and *a-bcd* are not unequal, that is, they are equal or equivalent to one another.

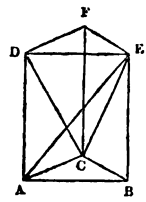


PROP. XV. THEOR.

EVERY prism having a triangular base may be divided into three pyramids that have triangular bases, and that are equivalent to one another.

Let there be a prism of which the base is the triangle ABC, and let DEF be the triangle opposite to the base: the prism ABC-DEF may be divided into three equivalent pyramids having triangular bases.

Draw AE, EC, CD, and because ABED is a parallelogram of which AE is the diameter, the triangle ADE is equal to the triangle ABE (Euc. 34. 1.): therefore the pyramid of which the base is the triangle ADE, and vertex the point C, is equivalent to the pyramid of which the base is the triangle ABE and vertex the point C (14. Prisms, &c.). But the pyramid of which the base is the triangle ABE and vertex the point C, that is the pyramid C-ABE, is equivalent to the pyramid C-DFE (14. Prisms, &c.), for they have equal bases, viz. the triangles ABC, DFE, and the same altitude, viz. that of the prism: therefore the three pyramids C-ADE, C-AEB, C-DFE, are all equivalent to one another. But these three pyramids make up the prism ABC-DFE: therefore the prism is divided into three equivalent pyramids.



Cor. 1. From this it is manifest that every pyramid is equivalent to the third part of a prism which has the same base and the same altitude with it; for if the base of the prism be any other figure than a triangle, it may be divided into prisms having triangular bases.

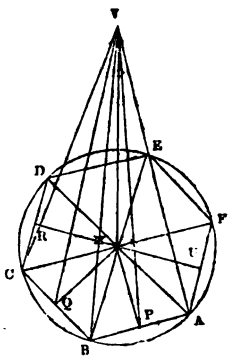
Cor. 2. Pyramids of equal altitudes are to one another as their bases; because the prisms upon the same bases and of the same altitude, are to one another as their bases. And, for a like reason, pyramids, that have equal or equivalent bases, are to one another as their altitudes.

Cor. 3. Pyramids are to one another in the ratio compounded of the ratios of their bases and altitudes; for the prisms which have their bases and altitudes equal to the bases and altitudes of the pyramids, are to one another in the ratio compounded of the ratios of their bases and altitudes.

PROP. XVI. THEOR.

If an upright pyramid have for its base a regular polygon, the convex surface of that pyramid is equivalent to the rectangle contained by a perpendicular let fall from the vertex of the pyramid upon one of the sides of the base and a line equal to half the perimeter of the base.

Let ABCD, &c., be a regular polygon, constituting the base of the pyramid, and let VZ be the axis, Z being the centre of a circle described about the base.



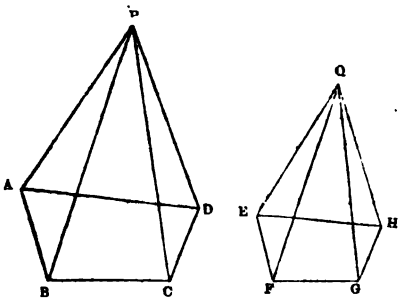
In the plane of the base let fall ZP, ZQ, &c. perpendicularly on the sides of the polygon; also draw VP, VQ, &c. Since the sides AB, BC, are equal to one another, ZP, ZQ, &c. will also be equal to one another (Euc. 14. 3.), and AB, BC, &c. will be bisected in P, Q, &c. (Euc. 3. 3.) Then in the triangles VZP, VZQ, &c., ZP, ZQ, &c. are equal to one another, VZ is common, and all the angles VZP, VZQ, &c. are right angles: therefore VP, VQ, &c. will be equal to one another. Again, since VA is equal to VB, AP to PB, and that VP is common, VP is perpendicular to AB (Euc. 8. 1.); and, in like manner, VQ, &c. are at right angles to BC, &c. But the triangle VAB is equivalent to the rectangle contained by VP and the half of AB; and, since all the perpendiculars VP, VQ, &c., are equal to one another, the sum of all the triangles, that is the convex surface of the pyramid, is equivalent to the rectangle contained by one of the perpendiculars, and a line equal to half the perimeter of the polygon constituting the base. Q. E. D.

Cor. The convex surfaces of upright pyramids having equal altitudes, and whose bases are regular polygons of an equal number of sides, are to one another as the perimeters of the bases: and if the upright pyramids have equal bases, their convex surfaces are to one another as their altitudes.

PROP. XVII. THEOR.

If two pyramids have their bases similar and similarly situated, and have likewise two of their other planes similar and similarly situated, and alike inclined to the planes of their bases, they are similar to one another.

Let P-ABCD and Q-EFGH be two pyramids, the bases of which are the similar and similarly situated figures ABCD and EFGH; and let two of their other planes, APB and QEF, be similar and similarly situated, and alike inclined to ABCD and EFGH: the pyramid P-ABCD is similar to Q-EFGH.



Because the triangle PAB is similar to the triangle QEF, the angle ABP is equal to the angle QFE. And because the bases of the pyramids are similar, the angle ABC is equal to the angle EFG: wherefore the solid angles at B and F, have two of their plane angles equal, each to each, and the planes in which those angles are, have like inclinations to one another: consequently the remaining plane angles PBC and QFG are equal to one another, and the planes PBC and FQG have like inclinations to the planes APB and EQF, and also to the planes ABCD and EFGH (23. Planes). Again, because the triangles APB and EQF are similar,

PB : BA :: QF : FE;

and, because the bases of the pyramids are similar,

BA : BC :: FE : FG;

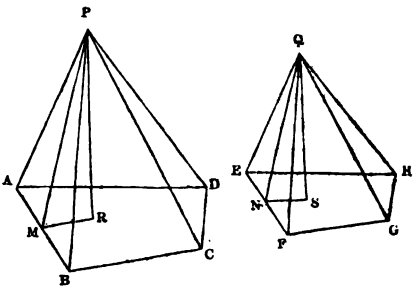
therefore, ex æquali, PB : BC :: QF : FG; that is, in the triangles BPC and FQG, the sides about the equal angles PBC and QFG are proportionals: wherefore the triangle BPC is similar to FQG (Euc. 6. 6.). And because the triangles BPC and FQG are similar, and have their planes alike inclined to the bases of the pyramids, it may be proved in the same manner that the triangles CPD and GQH are similar, and have their planes alike inclined to the bases of the pyramids, and likewise to the planes BPC and FQG; and so on, for all the other homologous planes of the solids. Wherefore the pyramids are similar, since they are contained by the same number of planes which are similar, and similarly situated, and alike inclined to one another. Q. E. D.

PROP. XVIII. THEOR.

THE altitudes of similar pyramids have to one another the same ratio that the homologous sides have.

Let P-ABCD and Q-EFGH be two similar pyramids, of which PR and QS are the altitudes, viz. the perpendiculars drawn from P and Q upon the planes ABCD and EFGH: the altitude PR is to the altitude QS, as any side of the pyramid

P-ABCD to the homologous side of the pyramid Q-EFGH.



Let PAB and QEF be homologous planes; draw RM perpendicular to AB, and SN to EF; join P, M and Q, N. Because PR is perpendicular to the plane ABCD, the plane PRM is also perpendicular to ABCD (18. Planes.). And because BM is in one of two perpendicular planes at right angles to their common section RM, it is perpendicular to the other plane PRM (2. Def. Planes.): wherefore the straight line BM is perpendicular to PM (1. Def. Planes.). And because PM and MR are both perpendicular to BM, the angle PMR is the inclination of the plane PAB to the plane ABCD (4. Def. Planes.). In like manner, it is shown that the angle QNS is the inclination of the plane QEF to the plane EFGH. But, because the pyramids are similar, the homologous planes PAB and QEF are alike inclined to the bases ABCD and EFGH: wherefore the angle PMR is equal to QNS, and consequently the right-angled triangle PMR is equiangular to the right-angled triangle QNS. And, because the planes PAB and QEF are similar to one another, the angle PAB is equal to QEF, and consequently the right-angled triangle PAM is equiangular to the right-angled triangle QEN. Again, because the triangles PMR and QNS are equiangular,

RP : PM :: SQ : QN;

and, because the triangles PAM and QEN are equiangular,

MP : PA :: NQ : QE;

ex æquali, RP : PA :: SQ : QE

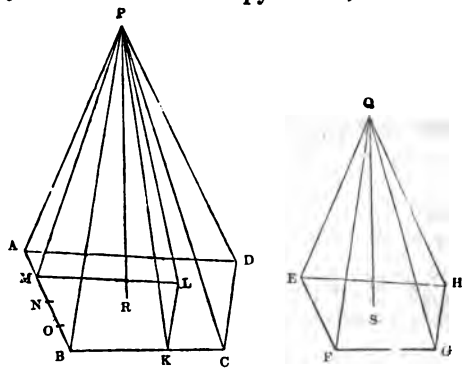
and, alternately, RP : SQ :: PA : QE.

But, because the solids have all their bounding planes similar to one another, each to each, it is easy to prove that PA has to QE, the same ratio that any side of the pyramid P-ABCD has to the homologous side of the pyramid Q-EFGH. Wherefore the altitude PR has to the altitude QS the same ratio that any two homologous sides of the pyramids have to one another. Q. E. D.

PROP. XIX. THEOR.

SIMILAR pyramids are to one another in the triplicate ratio of their homologous sides.

Let P-ABCD and Q-EFGH be similar pyramids, of which AB and EF are homologous sides: the ratio of the pyramid P-ABCD to the pyramid Q-EFGH, is triplicate of the ratio of AB to EF.



Draw PR perpendicular to the plane ABCD, and QS perpendicular to EFGH. In BA take MB equal to EF, and upon MB construct the figure MBKL similar to either of the figures ABCD, EFGH, and therefore equal to EFGH: draw MP, KP, LP: make as AB to BM so BM to BN; and as AB to BM so BN to BO. Because the pyramids P-MBKL and Q-EFGH have equal bases, they are to one another as their altitudes (2. Cor. 15 Prisms, &c.); that is,

pyr. P-MBKL : pyr. Q-EFGH :: PR : QS.

But PR : QS :: AB : EF, or BM (18. Prisms, &c.) :: BN : BO: therefore, pyr. P-MBKL : pyr. Q-EFGH :: BN : BO.

Again, because the three straight lines AB, BM, BN are proportionals, and similar figures ABCD and MBKL are described on the first and second: therefore,

ABCD : MBKL :: AB : BN (Euc. 2. Cor. 20. 6.).

But, because the pyramids P-ABCD and P-MBKL have the same altitude,

pyr. P-ABCD : pyr. P-MBKL :: ABCD : MBKL (2. Cor. 15. Prisms, &c.):

wherefore pyr. P-ABCD : pyr. P-MBKL :: AB : BN.

And, since it has already been proved that

pyr. P-MBKL : pyr. Q-EFGH :: BN : BO;

therefore, ex æquali,

pyr. P-ABCD : pyr. Q-EFGH :: AB : BO. But the ratio of AB to BO is triplicate of the ratio of AB to BM, or of AB to EF (18. Def. Pr.): wherefore the ratio of the pyramid P-ABCD to the pyramid Q-EFGH is triplicate of the ratio of the side AB to the homologous side EF. Q. E. D.

Cor. Similar pyramids are to one another in the triplicate ratio of their altitudes. For take PR to QS as QS to X, and as X to Y; then PR is to Y in the triplicate ratio of PR to QS. But in the proposition, AB is to EF as EF to BN, and as BN to BO; and since (18. Prisms, &c.) the altitudes of similar pyramids have to one another the same ratio that the

homologous sides have, PR : QS :: AB : EF: whence all the above ratios are equal to one another, and ex æquali,

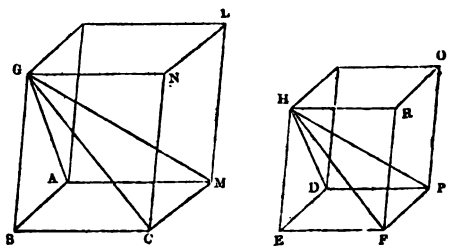
PR : Y : AB : BO.

But the pyramids are to one another as AB to BO: therefore they are to one another as PR to Y, that is, in a triplicate ratio of PR to QS, the altitudes.

PROP. XX. THEOR.

SIMILAR prisms are to one another in the triplicate ratio of their homologous sides.

Let BL and EO be two similar prisms; the prism BL has to the prism EO the triplicate ratio of that which the side BC has to the homologous side EF.



From G draw GA, GM, GC; and from H draw HD, HP, HF. Because the prisms are similar, the parallelogram GBCN is similar to the parallelogram HEFR (11. Def. Prisms, &c.): wherefore the angle GBC is equal to HEF, and GB is to BC as HE to EF (Euc. 1. Def. 6.); that is, the triangles GBC and HEF have their sides about the equal angles GBC and HEF, proportionals; these triangles are therefore similar to one another (Euc. 6. 6.). And because the prisms are similar, the base ABCM is similar to the base DEFP, and the plane GBC has the same inclination to the plane ABCM that the plane HEF has to the plane DEFP (11. Def. Prisms, &c.): wherefore the pyramids G-ABCM and H-DEFP are similar (17. Prisms, &c.), and they have to one another the triplicate ratio of that which BC has to EF (19. Prisms, &c.). But the prism BL is equivalent to three times the pyramid G-ABCM, and the prism EO, to three times the pyramid H-DEFP (1. Cor. 15. Prisms, &c.): wherefore, also, the prism BL has to the prism EO the triplicate ratio of that which the side BC has to the homologous side EF.

COR. Since cubes are similar prisms, they are to one another in the triplicate ratio of their homologous sides: and, hence, similar pyramids and prisms are to one another as the

cubes of their homologous sides. The cube formed on any line, as AB, is for the sake of brevity represented by AB^3.

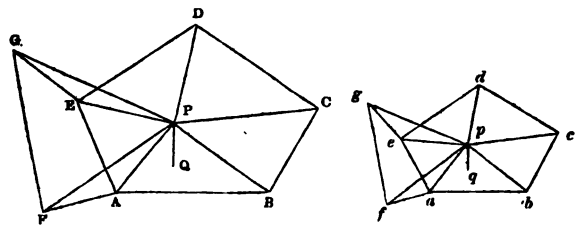
PROP. XXI. THEOR.

ALL similar solid figures are to one another in the triplicate ratio of their homologous sides.

The proposition has already been proved in pyramids (19. Prisms, &c.).

Let there be any other similar solid figures, viz. those of which the bases are the similar and similarly situated plane figures ABCDE and *abcde*; they are to one another in the triplicate ratio of their homologous sides.

Let AEGF and *aegf* be two homologous planes adjacent to ABCDE and *abcde*. Take any point P within the solid, and draw straight lines from P to all its angles: these straight lines, together with the edges of the solid, will form triangles, which will divide it into as many pyramids, having their com-



mon vertex at P, as it has bounding planes. Let AB and *ab* be any two homologous sides of the figures ABCDE and *abcde*; upon *ab* construct the triangle *apb* similar and similarly situated to the triangle APB; and let the plane *apb* be so placed as to have the same inclination to *abcde*, that the plane APB has to ABCDE; and from *p* let straight lines be drawn to all the angles of the solid upon the base *abcde*, which will therefore be divided into as many pyramids, having their common vertex at *p*, as the other solid is divided into.

First, the pyramids into which the solids are divided, are similar to one another, each to each, viz. those of which the bases are homologous planes. For the pyramid P-ABCDE is similar to the pyramid *p-abcde*, because their bases are similar, and two of their other planes, PAB and *pab*, are similar and alike inclined to their bases (17. Prisms, &c.). And because PAE and *pae* are homologous planes of two similar pyramids, they are similar to one another, and make equal

angles with the bases ABCDE and *abcde* (11. Def. Prisms, &c.): but the planes AEGF and *aegf* likewise make equal angles with the planes ABCDE and *abcde*: wherefore the plane APE has the same inclination to AEGF, that the plane *ape* has to *aegf*. The pyramid P-AEGF is therefore similar to the pyramid *p-aegf*, for the same reason as before (17. Prisms, &c.). And, in like manner, it may be proved that any two pyramids, of which the bases are homologous planes adjacent either to ABCDE and *abcde*, or to AEGF and *aegf*, are similar to one another. If, therefore, the planes of the solid upon the base ABCDE be taken in a series, the planes of the other solid will form a like series, those planes being homologous which have the same rank in the two series; and it may be proved successively, that any two pyramids of which the bases are homologous planes are similar to one another.

Next, the pyramids into which the solids are divided have the same ratio to one another, each to each. For the pyramid P-ABCDE has to the pyramid *p-abcde*, the triplicate ratio of their homologous sides AE and *ae* (19. Prisms, &c.); and the pyramid P-AEGF has to the pyramid *p-aegf*, the triplicate ratio of the same straight lines AE and *ae*: wherefore the pyramid P-ABCDE is to the pyramid *p-abcde*, as the pyramid P-AEGF to the pyramid *p-aegf*. And, in like manner, it may be proved that any two pyramids in the solid upon the base ABCDE, which are situated upon adjacent planes, have the same ratio to the two pyramids, similar to them, in the other solid: wherefore, taking the pyramids upon the two serieses of planes above-mentioned, any two pyramids next in order in one series will have the same ratio to the two pyramids of the same rank in the other series; and as the ratio of any pyramid in one series to the pyramid of the same rank in the other series will thus be equal both to the ratio of the two that precede them in order, and likewise to the ratio of the two that follow them in order, it is manifest that the pyramids in the two solids have the same ratio to one another, each to each; those in the solid upon the base ABCDE being the antecedents, and those in the solid upon the base *abcde*, the consequents.

And, lastly, as one antecedent is to its consequent, so are all the antecedents to all the consequents: wherefore, as the pyramid P-ABCDE is to the pyramid *p-abcde* so is the whole solid upon the base ABCDE to the whole solid upon the base *abcde*. But the pyramid P-ABCDE has to the pyramid *p-abcde* the triplicate ratio of their homologous sides AB and *ab* (19. Prisms, &c.): wherefore the solid upon the base

ABCDE has to the solid, upon the base *abcde*, the triplicate ratio of that which the side AB has to the homologous side *ab*. Q. E. D.

COR. 1. Since cubes are similar solid figures, they are to one another in the triplicate ratio of their homologous sides; and hence all similar solid figures are to one another as the cubes of their homologous sides.

COR. 2. If from any two corresponding points P and *p* in the two solids, perpendiculars as PQ, *pq*, be let fall on any two homologous planes, as ABCDE, *abcde*, the solids will be to one another in the triplicate ratio of the perpendiculars. For the pyramids P-ABCDE, *p-abcde* being similar, the perpendiculars PQ, *pq*, are to one another as the homologous sides: therefore those pyramids (Cor. 19. Prisms, &c.), and consequently; the whole solids are to one another in the triplicate ratio, or as the cubes of the perpendiculars.

COR. 3. Since the bounding planes of the two solids are similar figures, and have like inclinations to one another, it follows that the surfaces of the two solids are to one another in the duplicate ratio, or as the squares, of their homologous sides, and consequently of the perpendiculars PQ, *pq*.

ELEMENTS

OF

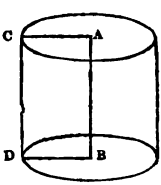
GEOMETRY.

ON THE CYLINDER, THE CONE, AND THE SPHERE.

DEFINITIONS.

I.

A CYLINDER is a solid figure described by the revolution of a right-angled parallelogram ABCD about one of its sides AB, which remains fixed.



II.

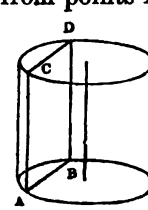
The axis of a cylinder is the fixed straight line AB, about which the parallelogram revolves.

III.

The bases of a cylinder are the circles described by the sides AC and BD of the parallelogram, which are perpendicular to the axis.

COR. 1. The axis of a cylinder is perpendicular to the planes of the bases (1. Def. Planes.).

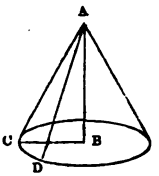
COR. 2. If a plane pass through any two lines AC, BD, drawn perpendicular to the base of a cylinder, from points A and B, in the circumference of the base, the section AD of the cylinder will be a rectangle parallel to the axis. For the two sides of the section coincide with two positions of the line, parallel to the axis, by whose revolution the surface of the cylinder is formed: they are, therefore, equal and parallel to the axis and to



one another; and consequently AB, CD are equal and parallel to one another. Also, since AC and BD are perpendicular to the base of the cylinder, they are perpendicular to AB and to CD; and, therefore, AD is a rectangle.

IV.

A cone is a solid figure described by the revolution of a right-angled triangle, ABC, about one of the sides, AB, containing the right angle, which side remains fixed.



V.

The axis of a cone is the fixed straight line, AB, about which the triangle revolves.

VI.

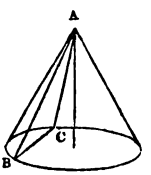
The base of a cone is the circle described by that side, BC, containing the right angle which revolves.

COR. The axis of a cone is perpendicular to the plane of its base.

VII.

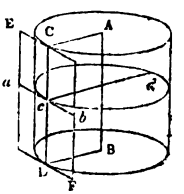
The side of a cone is any straight line, AC or AD, drawn from the vertex, A, to the circumference of the base.

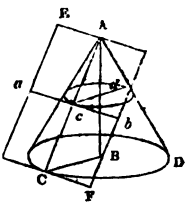
COR. If a plane pass through the vertex A of a cone, and any two points BC in the base, the section ABC of the cone will be an isosceles triangle; for the sides of the section coincide with two positions of the revolving hypotenuse, by which the surface of the cone is formed.



N. B. Every section parallel to the base of a cylinder or cone is a circle, because such a section coincides with a plane figure formed by the revolution of a line about the axis, to which it is perpendicular.

Also a plane EF, which is at right angles to that of the rectangle AD, by whose revolution a cylinder is described, if it pass through that side, CD, of the rectangle, which is parallel to the axis AB, shall touch the cylinder in that line through which it passes: and a plane EF, which is at right angles to that of the triangle ABC, by whose revolution a cone is described, if it pass through the side AC, which is oblique to the axis, shall touch the cone in that line





through which it passes. For in each case, if a plane parallel to the base cut both the said plane EF and the solid of revolution, the intersection *ab* with the former will be a straight line, and with the latter will be a circle *cd*: but each straight line will be perpendicular to a diameter *cd*, of the circle at the point of contact, and it will therefore touch the circle in one point only (Euc. 16. 3.); therefore a line joining all the points of contact, that is, the line CD or AC, through which the first plane passes, will be that in which the plane touches the cylinder or cone.

VIII.

A frustum of a cone is a portion CD*cd* (in the last figure) situated between the base CD and a section *cd* parallel to the base.

IX.

Similar cylinders and cones are those which have their axes and the diameters of their bases proportional; that is, which are described by similar rectangles, or similar right-angled triangles.

X.

A sphere is a solid figure described by the revolution of a semicircle about a diameter of the circle, which diameter remains fixed.

XI.

The axis of a sphere is the fixed straight line about which the semicircle revolves.

XII.

The centre of the sphere is the middle of that straight line.

XIII.

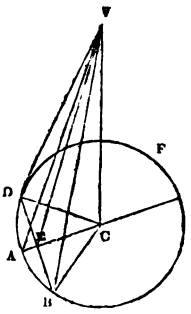
A diameter of a sphere is any straight line which passes through the centre, and is terminated both ways at the surface of the sphere.

PROP. I. THEOR.

EVERY section of a sphere is a circle.

If the cutting plane pass through the centre, the section is evidently a circle, because the sphere is described by the re-

will differ, in defect, from one of the oblique sides of the pyramid, or from the side of the cone, by a line less than any that can be assigned.



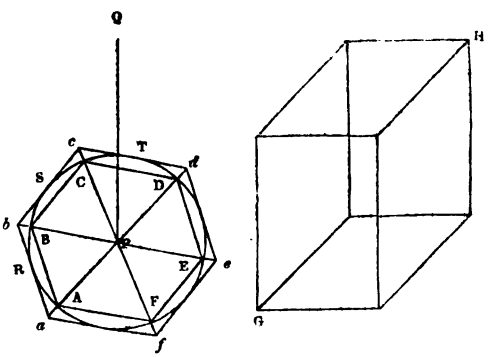
Let the circle BADF be the base of the cone, BD one side of the base of the pyramid, and V the vertex. Draw the radius CA at right angles to BD, consequently bisecting it in E: draw also VB, VD, and VE. Then, since by the nature of the cone the lines VA, VB, and VD are equal to one another; also since BD is bisected in E, and VE is common; the sides of the triangles VEB, VED are equal to one another; and consequently VE is perpendicular to BD.

In the right-angled triangles VCA, VCE, $VA^2 = VC^2 + CA^2$ (Euc. 47. 1.): also $VE^2 = VC^2 + CE^2$: taking equals from equals, $VA^2 - VE^2 = CA^2 - CE^2$. But, since (4. Circle) CE differs from CA by a line less than any that can be assigned, $CA^2 - CE^2$, and consequently the equivalent quantity $VA^2 - VE^2$, is less (4. Cor. A. Pr.) than any surface that can be assigned: hence, the difference between VA or VB and VE is less than any line that can be assigned. Q. E. D.

PROP. V. THEOR.

A CYLINDER is equivalent to a prism which has an equivalent base and an equal altitude.

Let the circle ABCE be the base of the cyclinder, and PQ its axis; also, let GH be the prism whose base and altitude are respectively equivalent and equal to those of the cylinder: the cylinder will be equivalent to the prism.

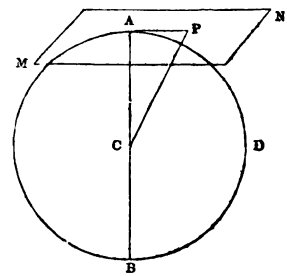


Let there be inscribed in the circle (7. Circle) a regular polygon which shall differ from it, in defect, by a magnitude

PROP. III. THEOR.

A PLANE which is at right angles to a diameter of a sphere, at one of the extremities of that diameter, touches the sphere.

Let MN be a plane at right angles to the diameter AB at the extremity A: it touches the sphere in that point only. For let a plane passing through AB cut the sphere in the great circle ADB, and the plane in the straight line AP. Let P be any point except A in that line; and, C being the centre of the sphere, draw CP: then, in the right-angled triangle CAP, because CAP is a right angle, APC is less than a right angle; and consequently (Euc. 19. 1.) CP is greater than the semidiameter CA: that is, P is on the exterior of the circle ADB. In like manner any point in the plane may be shown to be on the exterior of a great circle passing through A; and therefore the plane touches the sphere in the point A only. Q. E. D.



DEFIN. XVI.

A segment of a sphere is a portion cut off by a plane passing through the sphere.

DEFIN. XVII.

A zone of a sphere is a portion bounded by two parallel planes passing through the sphere, and the curve surface between the sections.

DEFIN. XVIII.

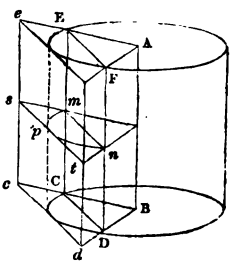
A sector of a sphere is a solid figure formed by the revolution of a sector of a circle about one of the straight lines containing it.

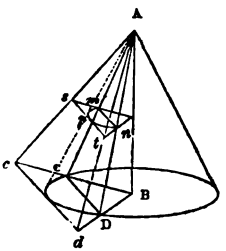
PRINCIPLES.

I.

If two planes ABC, ABD pass through the axis AB of a cylinder or cone and cut the surface of either (the lines of

section EC, DF on the cylinder being parallel to the axis, and the lines of section AC, AD on the cone being in directions from the vertex to the circumference of the base), the plane surface between the two lines of section will be less than the convex surface cut off by it from the cylinder or cone. For every straight line, as *mn*, which can be drawn on the plane parallel to the base, may be conceived to join the extremities of a circular arc, as *mpn*, on the surface of the cylinder or cone (the arc being in a plane which is parallel to the base and passes through the line *mn*): but (Princ. Circle) every such straight line will be less than the corresponding arc; therefore the whole plane surface will be less than the convex surface cut off by it.



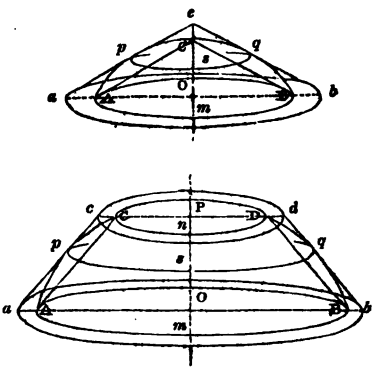


Again, if between the two planes ABc, ABd passing through the axis, and consequently perpendicular to the base of the cylinder or cone, a plane *ed* be in contact with the surface of the cylinder, or a plane A*cd* be in contact with the surface of the cone, the touching plane *ed* will be greater than the convex surface of the cylinder between the planes ED, *ed*; and the touching plane A*cd* will be greater than the convex surface of the cone, between the planes ACD, A*cd*. For every straight line, as *st*, drawn on the touching plane, parallel to the base of the cylinder or cone, and terminated on the two perpendicular planes ABc, ABd will (Princ. Circle) be greater than the circular arc, as *mpn*, on the surface of the cylinder or cone; and therefore the whole plane *ed* or A*cd* will be greater than the curve surface, on the cylinder or cone, between the planes ED, *ed*, or between the planes ACD, A*cd*.

Hence, the convex surface of a cylinder is greater than the convex surface of a prism inscribed within it, and less than that of a prism described about it. Also the convex surface of a cone is greater than the convex surface of a pyramid inscribed within it, and less than that of a pyramid described about it.

II.

If within any spherical segment ACB produced by the revolution of the figure C*p*AO about the axis CO, there be

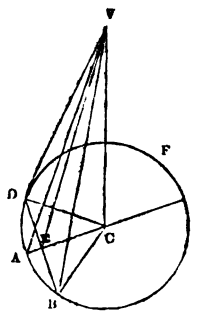


inscribed a cone CAB produced by the revolution of the triangle COA about the same axis, and if about it there be described a cone *cab* produced by the revolution of the triangle *cao* about the axis *c*O, the convex surface of the segment of the sphere will be greater than that of the cone CAB and less than that of the cone *cab*. Also, if within any spherical zone ABDC produced by the revolution of the figure POA*p*C about the axis PO, there be inscribed a conical frustum ABDC produced by the revolution of the rectilineal figure POAC about the same axis, and if about the same zone there be described a conical frustum *abdc* produced by the revolution of the rectilineal figure PO*ac* also about that axis, the convex surface of the spherical zone will be greater than the convex surface of the frustum ABDC and less than that of the frustum *abdc*. For every plane which may be imagined to pass through the solids of revolution, perpendicularly to their common axis will cut the surface of each solid in the circumference of a circle; and, except at the intersections A*m*B, C*n*D of the interior cone with the sphere, and at the contact *psq* of the exterior cone with the sphere (where the circumferences of the circles on the cone and sphere are coincident with one another), the diameter, and consequently the circumference, of the circle on the sphere will be greater than the diameter and circumference of the circle on the interior conical surface, and less than the diameter and circumference of the circle on the exterior conical surface. Therefore, &c.

PROP. IV. THEOR.

IF there be inscribed in an upright cone a pyramid having for its base a regular polygon formed by continually bisecting the circumference of the base of the cone till the arcs are less than any assignable lines, and joining the points of bisection; and if a perpendicular be let fall from the vertex of the cone upon one of the sides of the polygon, that perpendicular

will differ, in defect, from one of the oblique sides of the pyramid, or from the side of the cone, by a line less than any that can be assigned.



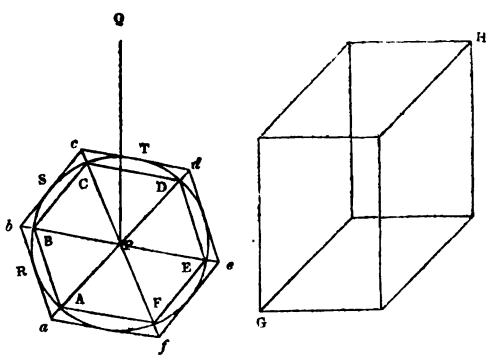
Let the circle BADF be the base of the cone, BD one side of the base of the pyramid, and V the vertex. Draw the radius CA at right angles to BD, consequently bisecting it in E: draw also VB, VD, and VE. Then, since by the nature of the cone the lines VA, VB, and VD are equal to one another; also since BD is bisected in E, and VE is common; the sides of the triangles VEB, VED are equal to one another; and consequently VE is perpendicular to BD.

In the right-angled triangles VCA, VCE, $VA^2 = VC^2 + CA^2$ (Euc. 47. 1.): also $VE^2 = VC^2 + CE^2$: taking equals from equals, $VA^2 - VE^2 = CA^2 - CE^2$. But, since (4. Circle) CE differs from CA by a line less than any that can be assigned, $CA^2 - CE^2$, and consequently the equivalent quantity $VA^2 - VE^2$, is less (4. Cor. A. Pr.) than any surface that can be assigned: hence, the difference between VA or VB and VE is less than any line that can be assigned. Q. E. D.

PROP. V. THEOR.

A CYLINDER is equivalent to a prism which has an equivalent base and an equal altitude.

Let the circle ABCE be the base of the cyclinder, and PQ its axis; also, let GH be the prism whose base and altitude are respectively equivalent and equal to those of the cylinder: the cylinder will be equivalent to the prism.



Let there be inscribed in the circle (7. Circle) a regular polygon which shall differ from it, in defect, by a magnitude

less than any surface that can be assigned; and let there be described about the circle a regular polygon of an equal number of sides, which shall differ from it, in excess, by a magnitude less than any surface that can be assigned. Next, let perpendiculars to the base of the cylinder be raised at all the angular points A, B, C, &c. of the inscribed polygon, and be terminated at the opposite extremity of the cylinder: all these perpendiculars will be in the convex surface of the cylinder; for the straight line parallel to PQ, by which the cylindrical surface is described, will successively coincide with them in its revolution about the axis PQ, and these perpendiculars will be the lateral edges of an upright prism supposed to be inscribed in the cylinder. Again, let perpendiculars to the plane of the base of the cylinder be raised at all the angular points *a*, *b*, *c*, &c. of the polygon described about the circular base, and be terminated in the plane which forms the opposite extremity of the cylinder: these perpendiculars will be the lateral edges of a prism described about the cylinder, and in contact with it in lines drawn perpendicularly to the base at the points R, S, &c., where the lines *ab*, *bc*, &c. touch the circle ABC.

Now the sum of the trapezoids between ABC, &c., *abc*, &c. being (7. Circle) less than any surface assignable, the space between the prism inscribed within, and the prism described about the cylinder, will (9, and Cor. 4. Prisms) be less than any solid assignable; therefore, also, the spaces between the cylinder and the two prisms will be less than any solids assignable. Again, since by hypothesis the base of the prism GH is equivalent to that of the cylinder (which differs in excess and defect from those of the prisms on ABC and *abc*, by magnitudes less than any surface assignable); and also since prisms which have equal altitudes are to one another as their bases; it follows that the prism GH will differ, in excess and defect respectively, from the prism inscribed in, and from the prism described about, the circle, by a magnitude less than any solid assignable: therefore, since both the prism GH (B) and the cylinder (A) differ, in excess and defect respectively, from the prism ($A_,$) inscribed within, and the prism (A') described about the cylinder, by a solid less than any that can be assigned, the cylinder (A) is, by the First General Theorem (Proportion) equivalent to the prism GH. Q. E. D.

Cor. Since cylinders are equivalent to prisms which have equivalent bases and equal altitudes, it may be inferred that all the relations which have been proved to subsist between prisms will hold good between cylinders. Thus, cylinders

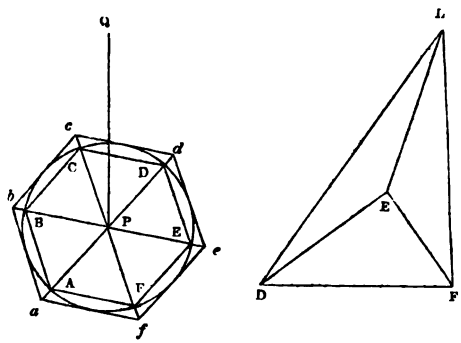
which have equal altitudes are to one another as their bases, or as the squares of the diameters of their bases; and those which have equivalent bases are to one another as their altitudes: also, cylinders are to one another in a ratio compounded of the ratios of their bases and altitudes; and similar cylinders are to one another in the triplicate ratio (or as the cubes) of their altitudes, or of the diameters of their bases.

PROP. VI. THEOR.

A CONE is equivalent to a pyramid which has an equivalent base and an equal altitude.

Let the circle ABCE be the base of the cone, and PQ its axis; and let L-DEF be a pyramid having a base and altitude equivalent and equal to those of the cone: the cone will be equivalent to the pyramid.

Let, as before, a polygon be inscribed within, and a polygon of an equal number of sides be described about, the base of the cone, the polygons being such that they differ from the circle, in defect and excess respectively, by magnitudes less than any surfaces assignable; and let lines be drawn from Q, the vertex of the cone, to all the angular points A, B, C, &c., *a*, *b*, *c*, &c.: these lines will form the edges of a pyramid inscribed within the cone, and of a pyramid described about it. The sum of the trapezoids between ABC, &c., *abc*, &c. being less (7. Circle) than any surface assignable, it follows that the space between the two pyramids (being equivalent to a pyramid formed of the surface between the polygons, as a base, and a line equal to PQ, as an altitude) will be less (1. Cor. 15, and Cor. 4. Prisms) than any solid assignable; therefore also the differences between the cone and the two pyramids will be less than any solids assignable.



But, by hypothesis, the base of the pyramid L-DEF is equivalent to that of the cone; also, since pyramids which have equal altitudes are to one another as their bases, it

follows that the pyramid L-DEF differs, in excess and defect respectively, from the pyramid inscribed within, and from the pyramid described about, the cone, by a magnitude less than any solid assignable. Therefore, since both the pyramid L-DEF (B) and the cone (A) differ, in excess and defect respectively, from the pyramid (A,) inscribed within, and the pyramid (A') described about, the cone, by a solid less than any that can be assigned, the cone (A) is, by the First General Theorem (Proportion), equivalent to the pyramid L-DEF. Q. E. D.

COR. From this, and from the preceding proposition, and also from 1. Cor. 15. Prisms, it follows that a cone is equivalent to one third of a cylinder having an equal base and an equal altitude. It is evident also that cones which have equal altitudes are to one another as their bases, or as the squares of the diameters of their bases; and that those which have equal bases are to one another as their altitudes. Again, cones are to one another in a ratio compounded of the ratios of their bases and altitudes; and similar cones are to one another in the triplicate ratio (or as the cubes) of their altitudes, or of the diameters of their bases.

PROP. VII. THEOR.

THE convex surface of a cylinder is equivalent to a rectangle contained by the altitude of the cylinder and a straight line equal to the circumference of the base.

Let a regular polygon (6. Circle) be inscribed within, and another, consisting of an equal number of sides, be described about, the base of the cylinder, so that their perimeters shall differ from one another, and consequently from the circumference of the base by lines less than any that can be assigned; and let lines be drawn from the angular points, perpendicularly to the bases, so as to form a prism inscribed within, and a prism described about the cylinder, as in Prop. 5.

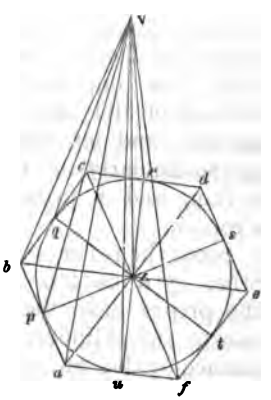
Since the perimeters of these polygons differ from one another by a line less than any that can be assigned, and that the prisms have the same altitude, viz. a line equal to the axis of the cylinder, the rectangles constituting the convex surface of the interior prism will differ (4. Cor. A. Pr.) from the rectangles constituting the convex surface of the exterior prism, by a surface less than any that can be assigned. But the convex surface of the cylinder is (1. Princ.) greater than the convex surface of the interior prism, and less than

that of the other; therefore it differs from them, in excess and defect respectively, by a surface less than any that can be assigned. Also, since the circumference of the base of the cylinder differs from the perimeters of the polygons, in excess and defect respectively, by a line less than any that can be assigned, the rectangle (B) contained by the axis of the cylinder and a line equal to the circumference of its base differs, in excess and defect respectively, from the surfaces (A), (A′) of the interior and exterior prisms, by a surface less than any that can be assigned: it follows, therefore, by the First General Theorem (Proportion) that the convex surface of the cylinder is equal to the rectangle contained by the axis of the cylinder and a line equal to the circumference of its base.

Cor. The convex surfaces of similar cylinders are to one another in the duplicate ratio (or as the squares) of their altitudes, or of the diameters of their bases.

PROP. VIII. THEOR.

If a regular polygon be described about the base of a cone, and lines be drawn from the vertex to the angular points of the polygon so as to constitute an upright pyramid, described about the cone, the convex surface of such pyramid is equivalent to the rectangle of the side of the cone and a straight line equal to half the perimeter of the polygon.



Let ab, bc, &c. be sides of the polygon, forming the base of the pyramid, and let them touch the circumference of the base of the cone in p, q, &c. Again, let v be the vertex of the cone, and draw vp, vq, &c.; also draw zp, zq, za, zb, &c. The lines zp, zq, &c. are (Euc. 18. 3.) perpendicular to ab, bc, &c., and (Euc. 12. 4.) these are bisected in p, q, &c. Therefore in the triangles zpa, zpb, zqb, zqc, &c., za, zb, zc, &c. are equal to one another. Again, in the triangles vza, vzb, vzc, &c., za, zb, zc, &c. are equal to one another, vz is common and at right angles to all these lines; therefore va, vb, vc, &c. are equal to one another. Then,

in the triangle, vab, va is equal to vb, ap to pb, and vp is common; therefore vp is at right angles to ab. In like manner vq, &c., all of which are equal to vp, the side of the cone, are at right angles to bc, &c.

Hence, all the triangles vab, vbc, &c. are equal to one another; but each of them is equivalent to the rectangle of the side of the cone and half of a side of the polygon described about the base; consequently the sum of all the triangles, or the convex surface of the pyramid, is equivalent to the rectangle of the side of the cone and half the perimeter of the base of the pyramid. Q. E. D.

Schol. The proposition is true whatever be the base of the pyramid which is described about the cone. For the planes vzp, vzq, &c. are always perpendicular to the base, and the sides of the figure forming the base are perpendicular to those planes, being perpendicular to zp, zq, &c. (the common section of those planes with the base), the sides of the figure are therefore perpendicular to vp, vq, &c. Therefore every triangle forming a face of the pyramid is equivalent to the rectangle contained by the side of the cone and half the line which forms the base of the triangle; and the convex surface of the pyramid is equivalent, &c.

PROP. IX. THEOR.

The convex surface of a cone is equivalent to a rectangle contained by its side, and a straight line equal to half the circumference of the base.

Let a regular polygon (6. Circle) be inscribed within, and another, consisting of an equal number of sides, be described about the base of the cone, such that the perimeters of the polygons shall differ from one another, and consequently from the circumference of the base, by a line less than any that can be assigned; and let lines be drawn from the vertex of the cone to the angular points of the polygons: these lines will constitute the edges of pyramids inscribed within and described about the cone.

Now (16. Prisms, &c.) the convex surface of the inscribed pyramid is equivalent to the rectangle contained by the perpendicular let fall from the vertex on one side of the inscribed polygon, and a line equal to half the perimeter of the base of the pyramid; and (8. Cylinder, &c.) the convex surface of the pyramid described about the cone is equivalent to the rectangle contained by a side of the cone and a line equal to

half the perimeter of the base of the pyramid. But the perpendicular let fall from the vertex on one side of the inscribed polygon differs from the side of the cone by a line less than any that can be assigned (4. Cylinder, &c.), and the perimeters of the bases of the polygons differ from one another also by a line less than any that can be assigned; therefore (4. Cor. A. Pr.), the convex surfaces of the two pyramids will differ from one another by a surface less than any that can be assigned. But again, the convex surface (A) of the cone is greater than the convex surface of the interior pyramid, and less than that of the exterior pyramid (1. Princ.); therefore it differs from them, in excess and defect respectively, by a surface less than any that can be assigned. Also, since the perpendicular let fall from the vertex on one side of the inscribed polygon is less than the side of the cone by a line less than any that can be assigned, and the circumference of the base of the cone differs from the perimeters of the polygons, in excess and defect respectively, by a line less than any that can be assigned, the rectangle (B) contained by the side of the cone and half the circumference of its base differs, in excess and defect, from the convex surfaces (A,) (A′) of the interior and exterior pyramids by surfaces less than any that can be assigned: it follows therefore, from the First General Theorem (Proportion), that the convex surface of the cone is equivalent to the rectangle contained by the side of the cone and half the circumference of its base. Q. E. D.

Cor. The convex surfaces of similar cones are to one another in a duplicate ratio (or as the squares) of their sides, of their altitudes, or of the diameters of their bases.

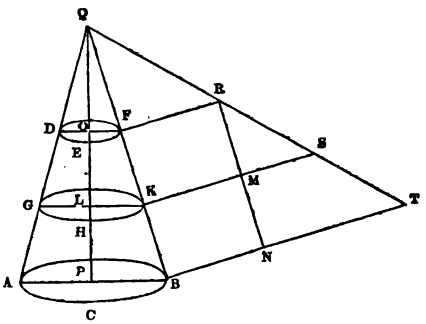
PROP. X. THEOR.

If a cone be cut by a plane parallel to the base, the part of the surface of the cone comprehended between the parallel planes is equivalent to a rectangle contained by the part of a side of the cone intercepted between the parallel planes, and a straight line equal to the circumference of another section of the cone, parallel to the base and to the first section, and at equal distances from them.

Let the circle ABC be the base and PQ the axis of a cone, and let the circle DEF be a section of the cone made by a plane parallel to the base; also let the circle GHK be another section likewise parallel to the base, and at equal distances from

the base and the section DEF: the part of the surface of the cone between the base and the section DEF is equal to the rectangle contained by BF and a straight line equal to the circumference of the circle GHK.

Let a plane drawn through the axis PQ meet the base and the sections parallel to it, in the straight lines PB, LK, OF, which lines are therefore parallel to one another (14. Planes). In the plane PQB draw BT perpendicular to QB, and equal to the circumference of the circle ABC; join Q and T, and draw FR, KS parallel to TB: also, through R draw RMN parallel to FB. Because the triangle QPB is similar to QFO, and the triangle QBT to QFR,



PB : BQ :: FO : FQ (Euc. 4. 6.),

also BQ : BT :: FQ : FR;

ex equali, PB : BT :: FO : FR,

and alternately, PB : FO :: BT : FR.

But PB : FO :: circum. PB : circum. FO (10. Circles, &c.); therefore BT : FR :: circum. PB : circum. FO. But BT = circum. PB; therefore FR = circum. FO. In the same manner it is proved that KS = circum. KL. Now the area of the triangle QBT = QB × $\frac{1}{2}$ BT = QB × $\frac{1}{2}$ circum. PB; the triangle QBT is therefore equal to the convex surface of the cone AQB (9. Cylinder, &c.). In like manner the triangle QFR is proved to be equal to the convex surface of the cone DQF. Wherefore the part of the surface of the cone comprehended between the base and the section DEF is equivalent to the trapezoid FRTB. Because the section GHK is equally distant from the base of the cone and the section DEF, FK or RM is equal to KB or MN, and RM = $\frac{1}{2}$ RN. Also, from the similarity of the triangles RMS, RNT, MS is equal to the half of NT (Euc. 4. 6.); hence, FR being equal to KM or BN, KS is equal to half the sum of the sides FR and BT of the trapezoid FRTB. Therefore, that trapezoid is equivalent (B. 1.) to the rectangle FB, KS; that is, the part of the convex surface of the cone comprehended between the base and the section DEF is equivalent to the rectangle contained by, &c. Q. E. D.

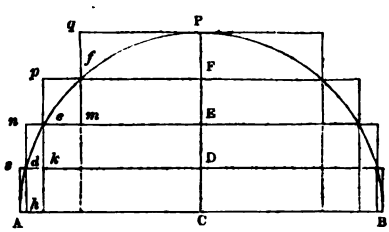
COR. 5. A. PROPORTION.

A prismatical or cylindrical solid contained by a given plane surface as a base, and an altitude less than any line that can be assigned, is a magnitude less than any solid that can be assigned. For it is that which remains when from a given prism, or cylinder having a given base, there is taken one which (having an equal base) is equal to half or more than half of the given solid; from the remainder one which (having an equal base) is equal to half or more than half of the remainder; and so on.

PROP. XI. PROB.

To form within a sphere a series of cylinders, which, together, shall differ from the sphere, in defect, by a magnitude less than the least solid that can be assigned; and to form a series of cylinders containing the sphere, which shall differ from the sphere, in excess, by a magnitude less than the least solid that can be assigned.

Let C be the centre, and APC be half of the semicircle by whose revolution about CP the sphere is described. By continual bisections divide the whole diameter, of which CP is the half, into equal parts, each less than the least assignable line, and through the points D, E, F, &c. of division, draw the lines D*d*, E*e*, F*f*, &c. perpendicular to CP: also through A, *d*, *e*, *f*, &c. draw the lines A*s*, *nh*, *ph*, *qm*, &c. parallel to CP. Then, if all the rectangles C*s*, C*d*, D*n*, D*e*, &c. in the semicircle revolve about PC, they will describe cylinders within and about the sphere.



Now, since all those cylinders have equal altitudes, it is manifest that the sum of all the annuli described on the surface of the hemisphere above AC by the revolution of the rectangles *sh*, *nk*, *pm*, &c., will be equal to the cylinder described by the revolution of C*s*; and the like will be true of the hemisphere below AC: but, since CD is less than the

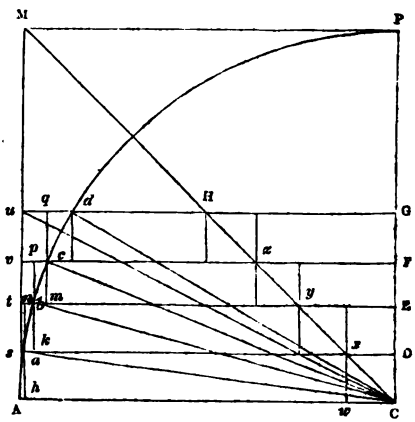
least line that can be assigned, the cylinder described by C*s* will (5. Cor. A. Pr.) be less than the least solid that can be assigned; and it follows that the sum of all the cylinders within the sphere will differ from the sum of all those which contain the sphere, by a magnitude less than the least solid that can be assigned. But the sphere is greater than all the cylinders within it, and less than all those about it; therefore, the sum of the interior cylinders will differ from the sphere, in defect, and the sum of those about the sphere, in excess, each sum by a magnitude less than the least solid that can be assigned. Q. E. F.

PROP. XII. THEOR.

A sphere is equivalent to two thirds of the cylinder described about it.

Let C be the centre, and APC be half of the semicircle by whose revolution about PC produced the sphere is described: also let PMAC be a square formed on AC, and constituting half of the rectangle by whose revolution about PC produced is described the cylinder which will circumscribe the whole sphere. Draw CM: then the triangle MPC will, by its revolution about PC, describe a cone, having C for its vertex, and circle MP for its base.

By continual bisections divide CP into equal parts, each less than the least assignable line, and through the points of division draw D*s*, E*t*, F*v*, &c. parallel to AC, intersecting the circle AP in *a*, *b*, *c*, *d*, &c., and CM in *x*, *y*, *z*, &c.: through *a*, *b*, *c*, &c. draw the lines *nh*, *pk*, *qm*, &c. parallel to PC and through *x*, *y*, *z*, &c. draw lines also parallel to CP as in the figure: the sum of the cylinders described by the revolution of the rectangles C*a*, D*b*, E*c*, &c., and the sum of those described by the revolution of the rectangles C*s*, D*n*, E*p*, &c. will (11. Cylinders) differ from the sphere, in defect and excess respectively, by magnitudes less than the least solid assignable; and it is easy



to perceive that the sum of the cylinders described by the revolution of the rectangles Ex, Fy, Gz, &c., and the sum of those described by the revolution of Dy, Ez, FH, &c. will, in like manner, differ, in excess and defect respectively, from the cone produced by the revolution of the triangle MPC, by magnitudes less than the least solid assignable. Draw Ca, Cb, Cc, &c.: then all the triangles CDa, CEb, CFc, &c. being right-angled at D, E, F, &c., in each of them, as in the triangle CFc (Cor. 9. Circle),

circle CF + circle Fc = circle Cc, or = circle Fv;

but, since Dx, Ey, Fz, &c. are parallel to MP, the triangles CDx, CEy, CFz, &c., are similar to MPC; and CP being equal to MP, CD = Dx, CE = Ey, &c., therefore

circle Fz + circle Fc = circle Fv.

But the cylinders described by the rectangles Ez, Ec, and Ev having the same altitude, viz. EF, on writing cylind. Ez for the cylinder described by the revolution of the parallelogram Ez, and a like abridgment for the other cylinders, it will follow (Cor. 5. Cylind.) that

cylind. Ez + cylind. Ec = cylind. Ev,

or cylind. Ec = cylind. Ev − cylind. Ez:

and the like is true of all the cylinders within the sphere. In a similar manner it may be shown that

cylind. Fy + cylind. Fb = cylind. Ft, or cylind. Ev,

or, cylind. Fb = cylind. Ev − cylind. Fy:

and the like is true of all the cylinders about the sphere.

Hence it follows that the sum of the cylinders within the hemisphere is equal to the difference between the cylinder described by the rectangle PA and the sum of those which contain the cone; also that the sum of the cylinders containing the hemisphere is equal to the difference between the cylinder described by the rectangle PA and the sum of those which are within the cone. Now the cone being less than the sum of the cylinders containing it, and greater than the sum of those within it, the difference in both cases being less than any assignable solid, it follows that the difference (A) between the cylinder described by the rectangle PA and the cone, is greater than the sum (A,) of the cylinders within the hemisphere, and that the same difference (A) is less than the sum (A′) of the cylinders containing the hemisphere, both the excess and defect being less than any assignable solid. But the hemisphere (B) is greater than the sum of the cylinders within it, and less than the sum of those which contain it, the difference in both cases being less than any solid assignable; therefore, by the First General Theorem (Proportion), (B) = (A), or the hemisphere, is equivalent to the

difference between the cone and the cylinder described by AMPC.

But the cone described by the triangle MPC is (Cor. 6. Cylind.) equivalent to one third of the cylinder described by AMPC; therefore the hemisphere is equivalent to two thirds of the same cylinder. The like is true of the hemisphere below AC; therefore the whole sphere is equivalent to two thirds of the circumscribing cylinder. Q. E. D.

COR. 1. Cylinders described about spheres are similar solids, since the rectangles by whose revolution they are described are similar to one another; therefore (Cor. 5. Cylind.) they are to one another in a triplicate ratio of their altitudes, or of the diameters of their bases; that is, of the diameters of the spheres. Therefore spheres are to one another in a triplicate ratio of their diameters.

COR. 2. The spherical sector produced by the revolution of any plane sector, as PC*d*, about PC, is equivalent to two thirds of that part of the cylinder described about the sphere, which is between the extremity P of the axis of revolution, and a plane passing through *d* perpendicularly to PC.

It follows from the proposition, that the spherical segment produced by the revolution of PG*d*, about the axis PC, is equivalent to the difference between the cylinder produced by the revolution of the rectangle MG and the conical frustum produced by the revolution of MHGP, about the same axis. Now, if to each of these equals be added the cone produced by the revolution of CG*d* about PC, the segment of the sphere and the last cone will make up the spherical sector produced by PC*d*; and

sector PC*d* + frustum MHGP = cylind. MG + cone GC*d*.

To each of these equals add the cone produced by HGC, observing that the frustum MHGP and the cone HGC make up the cone MPC; then,

sector PC*d* + cone MPC = cylind MG + cone GC*d* + cone HGC.

But circle GC + circle G*d* = circle G*u* (Cor. 9. Circle),

or, circle GH + circle G*d* = circle G*u*:

and if these circles be considered as the bases of cones whose common vertex is C, it will follow (Cor. 6. Cylind.) that

cone HCG + cone GC*d* = cone GC*u*:

also observing that cone MPC = one third of the cylinder AP, and that cone GC*u* = one third of the cylinder AG, it will follow that sector PC*d* + $\frac{1}{3}$ cylind. AP = cylind. MG + $\frac{1}{3}$ cylind. AG, or, subtracting $\frac{1}{3}$ cylind. AG from $\frac{1}{3}$ cylind. AP,

sector PC*d* + $\frac{1}{3}$ cylind. MG = cylind. MG;

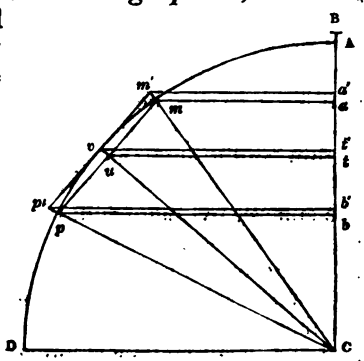
therefore the spherical sector produced by the revolution of PC*d* is equal to two thirds of the cylinder produced by the revolution of MG.

PROP. XIII. PROB.

To inscribe a conical figure in a sphere, and describe a similar conical figure about the sphere, such that their surfaces may differ from that of the sphere by magnitudes less than any surfaces that can be assigned.

Let AvD be half the semicircle, by whose revolution about AC the sphere is described: by continual bisections of the quadrantal arc AvD, and joining the points of division (3. Circle) let there be inscribed a quarter of a regular polygon whose sides shall be less than any lines that can be assigned; and let the straight line mp be one of the sides. Let $m'p'$ be the corresponding side of a like polygon described about the circle, touching the circumference in v, the middle point of the arc mvp.

Draw the semidiameter Cv intersecting mp in u, and draw ut, vt', ma, $m'a'$, pb, $p'b'$, all perpendicular to AC. Now if the sides mp, $m'p'$ be made to revolve about the axis AC, they will describe the surfaces of two frustums of cones between planes perpendicular to AC, the former within, and the latter on the exterior of the spherical surface described by the arc mvp; and (10. Cylind.) mp. circumf. ut, $m'p'$. circumf. vt' are respectively equivalent to the convex surfaces of those frustums. The sum of the convex surfaces formed by the revolution of all the sides of the polygon within the semicircle of which AvD is a quadrant, constitutes the superficies of a conical figure inscribed within the sphere; and the sum of the convex surfaces formed by the revolution of all the sides of the polygon about the same half circle constitutes the superficies of a like figure described about the sphere: but each frustum formed by the revolution of a line, as mp, is similar to the frustum formed by a corresponding line, as $m'p'$; the trapezoids $mabp$ and $m'a'b'p'$ being similar figures; therefore the whole solid figure which is inscribed in the sphere is similar to that which is described about it. Then, the perpendiculars Cu and Cv being homologous lines, $Cu^2 : Cv^2 ::$ surface of inscribed figure : surface of exterior figure; hence

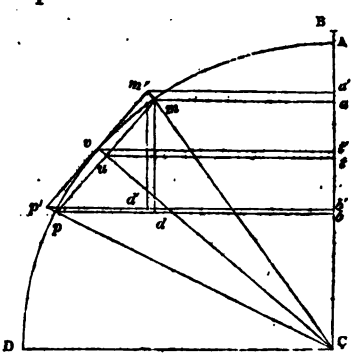


$Cu^2 : Cv^2 - Cu^2$ :: surface of inscribed figure : difference between the surfaces of the exterior and interior figures; or Cu^2 : surface of inscribed figure :: $Cv^2 - Cu^2$: difference between the surfaces of the exterior and interior figures. But (4. Circle) Cu differs from Cv by a line less than any that can be assigned; therefore $Cv^2 - Cu^2$ is (4. Cor. A. Pr.) less than any assignable surface, and consequently (2. Cor. A. Pr.) the difference between the surfaces of the two conical figures is less than any assignable surface. And the surface of the sphere, being between the conical surfaces, is greater than one and less than the other, it therefore differs from either, by a surface less than any that can be assigned. Q. E. F.

PROP. XIV. THEOR.

THE surface of a sphere is equivalent to the convex surface of a cylinder described about it.

Let AvD, as in the preceding proposition, be half the semicircle, by whose revolution about AC the sphere is described, and let mp, $m'p'$ be two corresponding sides of regular polygons, by whose revolution about AC there are described within and about the sphere similar conical figures, whose surfaces differ, in defect and excess respectively, from that of the sphere, by magnitudes less than any surface that can be assigned: also let fall perpendiculars on AC from the points m, m', u, &c., and draw md, $m'd'$ parallel to AC.



Then, because Cu is perpendicular to mp, the angles Cut and tum are together equal to a right angle: also in the triangle uCt, the angles Cut and uCt are together equal to a right angle; therefore the angle uCt is equal to the angle tum. But tum is equal to mpd (Euc. 29. 1.); therefore uCt is equal to mpd: and the angles at t and d being right angles, the triangles Cut and pmd are similar to one another; therefore (Euc. 4. 6.)

$$Cu : ut :: mp : md \ (= ab).$$

But the circumferences of circles are to one another as their semidiameters (10. Circle); therefore, and by equality of ratios,

circum. Cu : circum. ut :: mp : ab,
and hence, ab circum. Cu = mp circum. ut.
In like manner, $a'b'$ circum. Cv = $m'p'$ circum. vt'.

Now the sum of all the rectangles represented by mp circum. ut being, as in the preceding proposition, equivalent to the surface of the conical figure within the sphere, it follows that the sum of all the rectangles represented by ab circum. Cu; that is (since the sum of all the lines represented by ab is equal to the diameter of the sphere) 2 AC . circum. Cu (A,) is equivalent to the surface of the conical figure within the sphere, and (2 CB being the diameter of a circle described about the exterior polygon) the sum of all the rectangles represented by $a'b'$ circum. Cv; that is, 2 CB . circum. Cv (A′) is, in like manner, equivalent to the surface of the conical figure on the exterior of the sphere: these rectangles consequently differ from the surface of the sphere, in defect and excess respectively, by surfaces less than any that can be assigned. But, since Cv or AC exceeds Cu by a line less than any that can be assigned, 2 AC . circum. AC (B) exceeds 2 AC . circum. Cu (A,), by a surface less than any that can be assigned; and since the sum of all the heights represented by $a'b'$, that is, 2 CB, or 2 Cm', exceeds 2 AC by a line less than any that can be assigned; therefore 2 AC . circum. AC . (B) is less than 2 CB . circum. Cv (A′), by a surface less than any that can be assigned. It follows, therefore, from the First General Theorem (Proportion), that 2 AC . circum. AC is equivalent to the surface of the sphere. But 2 AC . circum. AC is equivalent to the convex surface of a cylinder described about the sphere (7. Cylind.); therefore the surface of a sphere is equivalent, &c. Q. E. D.

Cor. 1. The reasoning used above also shows that the convex surface of any segment or zone of the sphere is equivalent to the convex surface of a cylinder whose base is equal to a great circle of the sphere, and whose height is equal to that of the segment or zone.

Cor. 2. It follows also that the surfaces of spheres, being equivalent to the convex surfaces of the circumscribing cylinders, which are similar figures, will be to one another in a duplicate ratio of their diameters.

Cor. 3. The convex surface of a cylinder described about a sphere being equivalent to the rectangle contained by a diameter, and a line equal to the circumference; and a circle being equivalent to the rectangle contained by a semidiameter and a line equal to half the circumference; it follows that the convex surface of the cylinder is equivalent to four

times the circle: therefore the surface of a sphere is equivalent to four times the area of one of its great circles.

COR. 4. The whole surface of a cylinder described about a sphere (that is, the convex surface together with the circles at its extremities) is equivalent to six great circles of the sphere. But the surface of the sphere is equivalent to four great circles: therefore the surface of the sphere is to the whole surface of a cylinder described about it as 4 to 6, or as 2 to 3.

ELEMENTS

OF

GEOMETRY.

SPHERICAL GEOMETRY.

DEFINITIONS.

I.

The poles of a circle of the sphere are the extremities of that diameter of the sphere which passes through the centre of the circle perpendicularly to its plane.

COR. 1. All circles whose planes are parallel to one another have the same poles: and two great circles whose planes are perpendicular to one another pass through each other's poles: also if a great circle pass through the poles of any circle, it is at right angles to the plane of that circle.

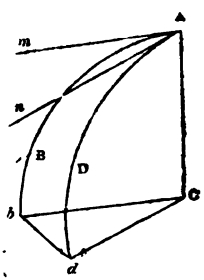
COR. 2. The pole of a circle is equally distant from every part of the circumference of that circle. This is proved in Prop. 2. Cylind.; where A is the pole of the circle N*pq*, and the straight lines AN, A*p*, A*q*, or the arcs AN, A*p*, A*q*, are respectively equal to one another.

Hence the pole of a great circle is at the distance of a quadrant from every part of the circumference of the circle: and two parallel circles intercept equal arcs on all great circles which pass through their poles.

II.

A spherical angle is the inclination of the planes of two circles of the sphere to one another.

Hence, the two angles formed when one circle falls obliquely upon another are equal to two right angles.



Cor. 1. A spherical angle, as BAD, is measured on a great circle whose plane is perpendicular to those of the two circles containing the angle, by an arc bd intercepted between the two circles; that is, a spherical angle is measured by an arc bd of a great circle, which is intercepted between the sides containing it, at the distance of a quadrant from the angular point A. For such an arc subtends, at the centre C of the sphere, an angle bCd, equal to the inclination of the planes of the circles containing the angle.

Cor. 2. If at the point A of a spherical angle straight lines, as Am, An, be drawn to touch the sides containing the angle, the plane angle mAn, contained by these straight lines, is equal to the spherical angle BAD: for the lines being in the planes of the circles, and perpendicular to AC, their common intersection, the angle between them is equal to the inclination of those planes; that is, to the spherical angle at A.

Cor. 3. The poles of two great circles which intersect one another are in the circumference of a great circle which is perpendicular to their common section; and since the pole of any great circle is at the distance of a quadrant from its circumference, it follows that the distance between the poles is equal to the measure of the spherical angle contained between those circles.

N. B. It is also the measure of the angle contained between two small circles having the same poles; since the planes of the small circles are then parallel to those of the two great circles.

III.

A spherical triangle is constituted on the surface of a sphere, by the arcs of three great circles which intersect one another.

If the three arcs are equal to one another, the triangle is said to be equilateral: if two arcs only are equal, it is called an isosceles triangle; and if one side is equal to a quadrant, it is called quadrantal.

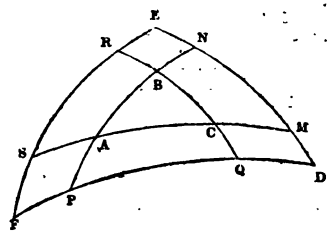
N. B. The difference between any acute angle and a right angle; or between a circular arc subtending such angle, and a quadrant of the circumference of the circle, is called the complement of that angle or arc. And the difference between any angle less than two right angles and the sum of two right angles, or between an arc subtending such angle,

and half the circumference of the circle, is called the supplement of such angle or arc.

PROP. I. THEOR.

If the angular points A, B, C of the spherical triangle ABC be the poles of three great circles, these circles will, by their intersections, form a spherical triangle DEF, which is said to be supplemental to the former; that is, the sides DE, DF, EF will be supplements of the measures of the opposite angles A, B, and C, and the sides AB, AC, BC will be supplements of the measures of the angles at D, E, and F.

Produce the sides AB, AC, and BC both ways on the surface of the sphere, till they intersect the sides of the exterior triangle DEF, in the points N, M, Q, P, &c. Then, first, since AB passes through A, the pole of DE, DE passes through the pole of AB (1. Cor. 1. Def. Sph. Geo.); and since BA passes through B, the pole of DF, DF passes through the pole of AB: therefore D, the intersection of DE and DF, must be the pole of AB. In like manner, E is the pole of AC, and F is the pole of BC.



Now, since D is the pole of AB, DN is a quadrant; and since E is the pole of AC, EM is a quadrant; therefore DN + EM, or DE + NM is equal to a semicircumference, and DE is the supplement of MN: but A being the pole of DE, MN (1. Cor. 2. Def. Sph. Geo.) is the measure of the angle at A; therefore DE is the supplement of the measure of the angle at A. In like manner, DF is the supplement of the measure of the angle at B, and EF is the supplement of the measure of the angle at C.

Next: since AB passes through A, the pole of DE, AN is a quadrant; and since BA passes through B, the pole of DF, BP is a quadrant; therefore AN + BP, or AB + NP is equal to a semicircumference, and AB is the supplement of NP: but D being the pole of AB, or PN, PN is the measure of the angle at D; therefore AB is the supplement of the measure of the angle at D. In like manner, AC is the supplement of the measure of the angle at E, and BC is the supplement of the measure of the angle at F. Q. E. D.

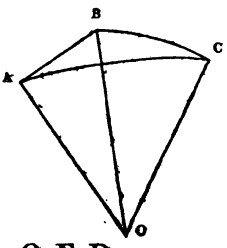
Hence, if one of the angles in the triangle ABC be a right angle, the triangle DEF is quadrantal.

N. B. Because D, E, and F are the poles of the sides of the triangle ABC, DEF has been called, with respect to the other, a polar triangle.

PROP. II. THEOR.

THE three sides of a spherical triangle, as ABC, are together less than the circumference of a circle.

Let O be the centre of the sphere, and draw OA, OB, and OC: then the arcs AB, AC, and BC will, respectively, be the measures of the plane angles AOB, AOC, and BOC; and (21. Planes) the sum of these angles is less than four right angles; therefore the sides of the triangle are together less than the measure of four right angles; that is, less than the whole circumference of a circle. Q. E. D.



PROP. III. THEOR.

THE three angles of a spherical triangle are greater than two right angles, and less than six.

For, in the triangle ABC (fig. to Prop. 1.), the measures of the angles at A, B, and C, together with the sides DE, DF, and EF of the supplemental triangle, are equal to three semicircumferences or six quadrants: but the sides of any spherical triangle are less (2. Sph. Geo.) than four quadrants; therefore the measures of the angles A, B, and C are together greater than two quadrants; that is, the angles are greater than two right angles.

Again: each of the interior angles at A, B, and C, with the adjacent exterior angle, are together equal to two right angles; therefore the three angles of the triangle, together with the three exterior angles, are equal to six right angles. Thus the angles A, B, and C of the triangle are less than six right angles. Therefore, &c. Q. E. D.

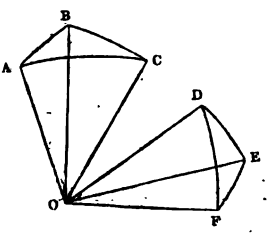
Hence, if one side of a spherical triangle be produced, the exterior angle is less than the sum of the two interior and opposite angles.

PROP. IV. THEOR.

IF the three sides of a spherical triangle, as ABC, be respectively equal to the three sides of another sphe-

rical triangle, as DEF, the angles opposite to the equal sides in each will be equal to one another.

Let O be the centre of the sphere, and draw the radii OA, OB, OC, OD, &c. Then, since AB is equal to EF, BC to DE, and AC to DF, and that these sides are respectively the measures of the plane angles AOB and EOF, BOC and DOE, &c., at the centre of the sphere; the three plane angles composing the solid angle belonging to one triangle, will be equal to the three plane angles composing the solid angle belonging to the other triangle. But, opposite to the equal plane angles, the inclinations of the planes are equal respectively (22. Planes), and these inclinations are equal to the angles of the spherical triangles; therefore, opposite to the equal sides, the angles of the one triangle are equal to those of the other. Q. E. D.



N. B. It may be observed that in this, and in each of the three following propositions, the surfaces of the triangles are equal to one another.

PROP. V. THEOR.

If the three angles of one spherical triangle are equal to those of another, the sides opposite to the equal angles are equal to one another.

For, let a supplemental triangle be formed about each of the given triangles: then, since the three sides of each supplemental triangle are the supplements of the measures of the angles of the triangle about which it is formed, the three sides of one of these triangles will be equal to the three sides of the other. Consequently, the angles of the one will be equal to the corresponding angles of the other: but the measures of these equal angles are supplements of the sides of the first triangles; therefore the sides of one of these triangles are equal to the sides of the other. Q. E. D.

PROP. VI. THEOR.

If two sides of one spherical triangle be equal to two sides of another, and the angles contained by those sides be equal, the remaining side and angles of the one triangle will be equal to the remaining side and angles of the other.

For, in the triangles ABC, DEF (fig. to Prop. 4.) if AB and AC be equal to FE and DF, the plane angles AOB, AOC will be equal to the plane angles FOE, DOF; also if the angle at A be equal to the angle at F, the inclination of AOB to AOC will be equal to that of FOE to DOF. Then (23. Planes) the plane angle BOC will be equal to DOE, and the remaining inclinations of the planes will be equal: but these plane angles are measured by BC and DE, and the inclinations are equal to the spherical angles at B and C, E and D; therefore BC is equal to DE, and the angles are respectively equal to one another. Q. E. D.

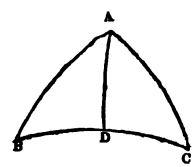
PROP. VII. THEOR.

If two angles of one triangle be equal to two angles of the other, and the sides adjacent to the equal angles in the two triangles are equal, the remaining angle and sides of one triangle will be equal to those of the other.

For let a supplemental triangle be formed about each triangle, as before: then two sides of one of these triangles will be equal to two sides of the other, and the angles contained between those sides will be equal; therefore, by the last proposition, the remaining side and the remaining angles of the one will be equal to the remaining side and the remaining angles of the other. But the supplements of these sides and angles are equal to the remaining angles and the remaining side in each of the given triangles; therefore those angles and sides are respectively equal to one another. Q. E. D.

PROP. VIII. THEOR.

The angles at the base of an isosceles spherical triangle are equal to one another.



Let ABC be a spherical triangle, of which the sides AB and AC are equal to one another. Bisect BC in D, and draw an arc of a great circle through A and D: then the three sides of the triangle ABD are equal to those of the triangle ACD; and consequently the angles opposite to the equal sides are equal: that is, the angles ABD and ACD are equal to one another. Q. E. D.

Hence, in an isosceles spherical triangle, if a great circle drawn through the angle contained by the equal sides bisect

the base, it will bisect that angle, and it will be perpendicular to the base.

PROP. IX. THEOR.

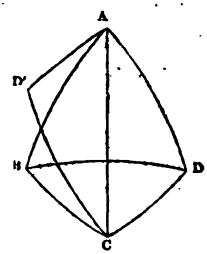
If the angles at the base of a spherical triangle be equal, the sides opposite to those angles will be equal.

For, let a supplemental triangle be formed; then the sides of the latter triangle which are opposite to the equal angles in the first triangle will be equal; and consequently, by the last proposition, the angles at its base will be equal. But these angles are supplements of the sides in the first triangle, which are opposite to the equal angles; therefore, those sides are equal to one another. Q. E. D.

PROP. X. THEOR.

If two right-angled triangles have a common hypotenuse, and have a side of the one equal to a side of the other, the remaining side and the remaining angles in each will be respectively equal to one another.

Let the triangles ABC, ADC, having the common hypotenuse AC, be right-angled at B and D, and have the side BC equal to the side CD. Draw a great circle through B and D; then in the triangle BCD, BC being equal to CD, the angles CBD and CDB will (8. Sph. Geo.) be equal to one another; taking these from the right angles at B and D, the remaining angles ABD, ADB will be equal to one another, and consequently (9. Sph. Geo.) AD will be equal to AB. Then, the three sides of the triangle ABC being equal to those of the triangle ADC, the angles opposite to the equal sides will be equal to one another. Q. E. D.



If two such triangles, as ABC, AD′C, be on the same side as the common hypotenuse AC (AD′ being the side which is equal to BC), the remaining side and angles in each will be equal to one another. For, if AD′C be placed in the position of CDA, it will coincide with the latter triangle: but this triangle has been proved to have its sides and angles equal to those of the triangle ABC; therefore the side and angles of CD′A will also be equal to those of ABC.

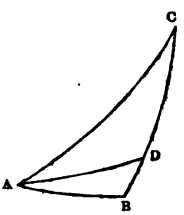
PROP. XI. THEOR.

Two sides of a spherical triangle are together greater than the third.

For the three sides are the measures of three plane angles which meet, and form a solid angle at the centre of the sphere; and of these angles, any two being greater than the third, it follows (20. Planes) that the two corresponding sides of the triangle are together greater than the third. Q. E. D.

PROP. XII. THEOR.

In any spherical triangle, the greater angle has the greater side opposed to it.



In the spherical triangle ABC let the angle BAC be greater than ACB, the side BC shall be greater than AB. For, make the angle CAD equal to the angle ACB; then will CD be equal to AD (9. Sph. Geo.): add to each the arc DB; then will AD + DB be equal to CB. But AD and DB together are greater than AB (11. Sph. Geo.); therefore CB is greater than AB. Q. E. D.

PROP. XIII. THEOR.

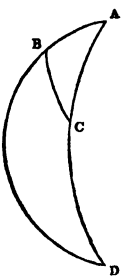
In any spherical triangle, the greater side has the greater angle opposed to it.

For, in the preceding figure, let CB be greater than AB; then, if the angle CAB be not greater than ACB it must be either equal or less. It cannot be equal to ACB, for then CB would be equal to AB (9. Sph. Geo.): neither can it be less, for then CB would be less than AB (12. Sph. Geo.); therefore the angle BAC must be greater than the angle ACB. Q. E. D.

PROP. XIV. THEOR.

In a spherical triangle, any two of the angles will together be equal to, greater than, or less than two right angles, as the opposite sides are together equal to, greater than, or less than a semicircumference.

Let ABC be a spherical triangle, and let the sides AB, AC be produced till they intersect one another, as at D. Then,



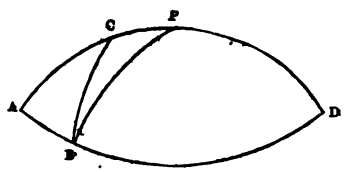
because the intersection of two great circles is a diameter of the sphere, ABD and ACD are each equal to a semicircumference. Now, first let AB and BC together be equal to a semicircumference, then BC will be equal to BD, and the angle BDC will be equal to BCD. But the angles at D and A are equal to one another, each of them being equal to the inclination of the planes of the two circles ABD, ACD to one another; therefore the angle BAC is equal to the angle BCD: add to each, the angle ACB: then BAC and BCA are together equal to BCA and BCD; that is, to two right angles. Next, let AB and BC be together less than a semicircumference; then BC will be less than BD, and the angle BDC will be less than BCD, that is, the angle BAC will be less than BCD: add to each the angle ACB; then BAC + BCA will be less than BCA + BCD, that is, less than two right angles. Therefore, &c. Q. E. D.

In like manner it may be proved, if AB + BC be greater than a semicircumference, that the sum of the angles BAC, ACB will be greater than two right angles.

N. B. In any spherical triangle, when two sides, or two angles, or a side and an angle, are each less, or each greater than a quadrant or a right angle, they are said to have the same affection; but if one be greater and the other less, they are said to have a contrary affection.

PROP. XV. THEOR.

In any right-angled spherical triangle, the sides about the right angle have the same affection as the opposite angles.

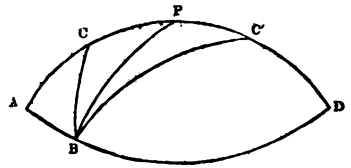


Let ABC be a spherical triangle, right-angled at A. Produce AC and AB till they meet, as in D; then ACD and ABD will each be a semicircumference, and if ACD be bisected in P, that point will be the pole of ABD. Draw the great circle PB; then PB, PA, and PD will be quadrants, and ABP will be a right angle. Consequently, while the angle ABC is less than ABP, that is, than a right angle, the side AC will be less than AP, that is, than a quadrant; and when ABC is greater than ABP, AC will be greater than AP. Therefore, &c. Q. E. D.

PROP. XVI.

In any right-angled spherical triangle, the hypotenuse is less or greater than a quadrant, according as the sides about the right angle are of the same, or of contrary affection.

Let ABC be a triangle, right-angled at A: produce AC and AB to intersect one another in D; bisect ACD in P, and join P, B: then P will be the pole of ABD; PB, PA, and PD will each be a quadrant, and the angle ABP will be a right angle.



First, let AB and AC be each less than a quadrant; then BC will be less than a quadrant. For, since in right-angled triangles, the sides have the same affection as the opposite angles, the angle ACB is less than a right angle; and therefore the angle BCP is greater than a right angle. For a like reason the angle APB is less than a right angle: thus the angle CPB is less than PCB, and therefore (12. Sph. Geo.) BC is less than BP: but BP is a quadrant; therefore BC is less than a quadrant.

If two sides of a spherical triangle are each greater than a quadrant, the hypotenuse will be less than a quadrant. For BD and CD are two sides of a triangle, right-angled at D; then BC will be the hypotenuse, and BC has been proved to be less than a quadrant.

Next, let ABC′ be the triangle, right-angled at A, and let the sides AB, AC′ about A have a contrary affection, AB being less, and AC′ greater than a quadrant. Since the sides of right-angled triangles have the same affection as the opposite angles, APB and AC′B have the same affection as AB; that is, each of them is less than a right angle; therefore the angle BPC′ is greater than a right angle, and consequently (12. Sph. Geo.) BC′ is greater than BP: but BP is a quadrant; therefore the hypotenuse BC′ is greater than a quadrant. Therefore, &c. Q. E. D.

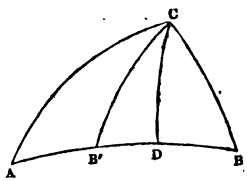
Cor. 1. If the angles at B and C have the same, or a contrary affection, the hypotenuse will be less, or greater than a quadrant: for the opposite sides AC and AB will have the same, or a contrary affection; therefore, &c.

Cor. 2. If one side and the adjacent angle have the same, or a contrary affection, the hypotenuse will be less or greater than a quadrant: for the angle will have the same affection

as the opposite side; and therefore, in this case also the two sides will have the same or a contrary affection.

PROP. XVII. THEOR.

THE angles at the base of any spherical triangle have the same or a contrary affection, according as the perpendicular let fall from the other angle, is within or on the exterior of the triangle.



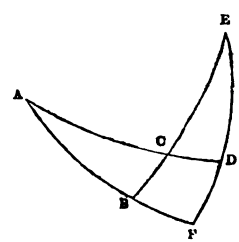
Let ABC be a spherical triangle, and let the perpendicular CD fall within it: the angles at A and B will have the same affection. For in the right-angled triangles ACD, BCD, each of those angles will have the same affection as CD; therefore one of them will have the same affection as the other.

Again, let AB'C be the triangle, and let the perpendicular CD fall outside of the triangle: the angle at A, and the angle CB'D will have the same affection as CD: but the affection of the angle AB'C is contrary to that of CB'D; therefore the angles CAB' and AB'C have a contrary affection. Therefore, &c. Q. E. D.

N. B. A spherical triangle whose sides and angles are complements of the sides and angles of another right-angled spherical triangle is said to be a complemental triangle.

PROP. XVIII. THEOR.

LET ABC be a spherical triangle right-angled at B; and let the sides AB and AC be produced to F and D, so that AF and AD may each be a quadrant; produce BC indefinitely, and through F and D draw a great circle meeting BC, produced, in E: the triangle CDE is complemental to the triangle ABC.



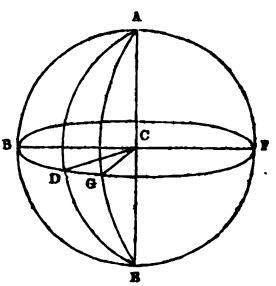
Because AD and AF are each equal to a quadrant, A is the pole of FD, and the angles at F and D are right angles: then, since the angles at B and F are right angles, the circles BCE, FDE are perpendicular to ABF; and E, their point of meeting, is the pole of ABF; therefore EB and EF are each quadrants. Now AD being a quadrant, CD is the complement of AC, and BE being a quadrant,

CE' is the complement of BC. Again, AF being a quadrant, BF is the complement of AB; but BF is the measure of the angle at E; therefore the angle at E is the complement of AB. Lastly, EF being a quadrant, ED is the complement of DF: but DF is the measure of the angle at A; therefore ED is the complement of the angle at A. Q. E. D.

PROP. XIX. LEMMA.

THE spherical surface contained between the halves of two great circles, is equal to a rectangle contained by a diameter of the sphere, and a line equal to the length of an arc which measures the inclination of the planes of the two circles.

Let ABE, ADE, AGE be great circles intersecting one another in the common diameter AE: let C be the centre of the sphere, and let BDF be a great circle perpendicular to AE: draw CB, CD, and CG: then BD and DG are arcs which measure the spherical angles BAD, DAG. Now all the arcs AB, AD, AG, EB, &c., being quadrants, it follows that when the arcs BD and DG are equal to one another, the quadrantal triangles ABD, ADG, BED, DEG, and consequently the lunaric surfaces between ABE and ADE, ADE and AGE are equal to one another. Hence, such lunaric surfaces are to one another as the arcs BD and DG which determine their breadths; and consequently (9. Propor.) the whole surface of the sphere is to any one lunaric surface (as that between ABE and ADE), as the circumference of the circle BDF is to the arc (as BD), which measures the angle (BAD or BCD) between the planes of the circles which bound the surface; that is, if AE represent the diameter of the sphere, AE. circumf. AE will represent its surface; and if arc A be put to represent the arc which measures any spherical angle, as BAD,



Circumf. AE is to AE. circumf. AE as arc A is to the surface of the sphere between ABE and ADE, or (Cor. 1. Propor.),

1 : AE :: arc A : surface between ABE and ADE.

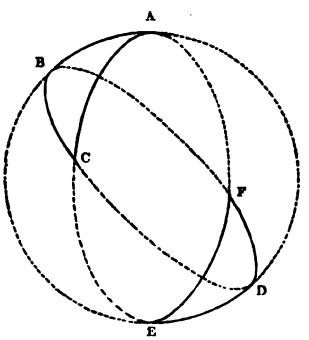
That is (Euc. 16. 6.), the lunaric surface is equal to the rectangle contained by AE and a line equal to the arc A. Q. E. D.

PROP. XX. THEOR.

THE surface of a spherical triangle is equal to a rectangle contained by a semi-diameter of the sphere, and a straight line equal to the excess of the arc (of a great circle) which measures the sum of its three angles, above the semicircumference of such circle.

Let ABC be a spherical triangle formed by the arcs of three great circles of the sphere which intersect one another at A, B, and C on one side of the sphere, and at E, D, and F on the opposite side.

Since great circles intersect one another in a diameter of the sphere, ABE, BED, BCD, CDF, ACE, and CEF are semicircumferences, consequently equal to one another; therefore, taking away parts which are common, there remain AB equal to ED, BC equal to DF, and AC equal to EF: that is, in the two triangles ABC, DEF the sides of the one are equal to the sides of the other; therefore (4. Sph. Geom.) the triangles are equal to one another. Let P represent the spherical surface BCE, Q represent the space CED, and R the space ACD: also,



let AE be the diameter of the sphere, and let arc A, arc B, arc C, represent the arcs which measure the spherical angles at A, B, and C respectively. Then (19. Sph. Geo.) the spherical surface between the semicircumferences ABE and ACE; that is, ABC + P is equal to the rectangle AE. arc A; the surface between CEF and CDF; that is, DEF + Q, or ABC + Q is equal to AE. arc C; and the surface between BCD and BAD; that is, ABC + R is equal to AE. arc B: therefore 3 ABC + P + Q + R is equal to the rectangle contained by AE and a line equal to the sum of the arcs A, B, and C. But ABC + P + Q + R is equal to half the surface of the sphere; that is, to the rectangle AE. $\frac{1}{2}$ circumf. AE; therefore,

2 ABC + AE. $\frac{1}{2}$ circumf. AE = AE (arc A + arc B + arc C).

Consequently, taking equals from equals,

2 ABC = AE (arc A + arc B + arc C) − AE. $\frac{1}{2}$ circumf. AE,

or ABC is equal to the rectangle contained by $\frac{1}{2}$ AE, and by a line equal to the difference between the half circumference

and the sum of the arcs which measure the angles at A, B, and C. Q. E. D.

COR. The surface of a spherical polygon may be obtained on dividing it into triangles by drawing great circles through the angular points, and adding together the expressions for the surfaces of all the triangles. It will be found that the surface of the polygon is equal to a rectangle contained by a line equal to an arc which measures the sum of all the angles of the polygon and a line equal to that multiple of the half circumference of a great circle which is expressed by the number of sides of the polygon diminished by 2.

THE END.

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